Solving Bewley Models with bilateral wage bargaining

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Abstract

Search-and-Matching models with incomplete markets à la Bewley appear challenging to solve, as standard wage bargaining protocols imply workers' wages to depend on their wealth. In fact, I demonstrate that they can be analyzed quickly by building on the Endogenous Grid Method (EGM), particularly if one uses a novel *Match-Integrated Endogenous Grid Method* (MIEGM): Its key feature is that it obtains worker- and firm value functions jointly instead of solving an outer functional fixed point problem. I show that this fast algorithm can be applied to a variety of models, including set-ups with endogenous separations or intensive margin labor supply. Additionally, the joint solution procedure facilitates studying aggregate shocks and transition dynamics using recent Sequence Space methods.

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1 Introduction

Two of the most successful workhorse models in modern macroeconomics seem to offer natural complementarities: On the one hand, incomplete markets frameworks à la Bewley-Huggett-Aiyagari (BHA) improved our understanding of individuals' consumption and savings decisions in the presence of uninsurable income- and unemployment risk. On the other hand, Search-and-Matching (SaM) models à la Diamond-Mortensen-Pissarides (DMP) provide an equilibrium theory of labor market frictions driving such risks.

Yet, combining both seems challenging: A key feature of DMP models is that once a match between a worker and a firm is formed, the two parties need to agree on a wage. This is typically modelled by splitting the joint surplus according to axiomatic bargaining theories such as the Nash (1950) bargaining solution. Under incomplete markets, the match surplus depends on the workers' wealth, as intuitively, unemployment is less painful for a rich worker than for a poor one. Thus, the model gives rise to a wealth-dependent wage schedule that needs to be solved for.

To do so, Krusell, Mukoyama, and Şahin (2010) - henceforth KMS - and subsequent authors proposed algorithms searching iteratively for a functional fixed point: Repeatedly solve worker- and firm value functions for different wage schedules until one is found that satisfies the bargaining solution for all wealth levels on a grid. These methods can indeed be costly, as they rely on solving the worker- and firm value functions for different wage schedules many times using Value Function Iteration (VFI). This is compounded by convergence of the outer wage loop being sensitive to various numerical specifications such as the grid structure or updating rules.

As I will show, the first issue can be addressed by using a version of the Endogenous Grid Method (EGM) (Carroll, 2006) instead of VFI. While this requires only straightforward modifications compared to the standard consumption-savings-problem, it has, to the best of my knowledge, not been used for BHA-DMP models with wealth-dependent wages. The key innovation of this paper, however, is the insight that instead of solving for the wage schedule in an *outer loop* as in the previous algorithms, one can also solve for it in the same *inner loop* as the value functions. For previous (guesses of) worker- and firm value functions, find the wage schedule that satisfies the bargaining solution, and use this wage schedule to update the value functions. Practically, instead of solving two simpler recursive problems many times, one solves a more complex problem only once.

While it is not immediately obvious that the latter approach should be beneficial, I demonstrate that it is indeed faster and more robust for a benchmark model when using it with my novel *Match-Integrated Endogenous Grid Method* (MIEGM), which extends EGM to such a joint recursive problem. Another benefit of considering the combined formulation is that, using the parlance of Auclert et al. (2021), workers' and firms' value functions are combined into the same "Block". In turn, it becomes possible to use their fast Sequence

Space methods for studying aggregate shocks and transition dynamics for BHA-DMP models with bilateral bargaining.[1]

To present my method, I apply it to known benchmark frameworks: I start with a set-up as in KMS, for which I will compare my method to previous algorithms. However, given the relative simplicity of that framework, I also demonstrate that my algorithm can be applied to richer environments, such as a setting with endogenous separations and a model with intensive margin labor supply as in Nakajima (2012). Both of these cases require some simple extensions.

It is worth noting that all these frameworks analyze settings with infinitely lived agents and Nash bargaining every period, i.e. no individual level wage stickiness. MIEGM should be equally applicable to models with finite horizons, such as set-ups with a life cycle dimension.[2] I also expect it to accommodate other bargaining protocols, as long as their wage outcome can be characterized through a few equations involving firm- and worker value functions. Incorporating idiosyncratic wage stickiness would require a more involved extension including the current wage as state variable into firm- and worker value functions. While these appear feasible in principle, the challenging task of their implementation goes beyond the scope of this paper.

1.1 Related Literature

In addition to the already mentioned papers, my work naturally relates to other papers studying BHA-DMP models with bilateral bargaining. Recent examples include Setty and Yedid-Levi (2021), Pizzo (2023) or Oliveira (2024), who focus on stationary environments using variants of the *outer loop* algorithms. While the recent literature on Heterogeneous Agents New Keynesian (HANK) models also studies set-ups with DMP labor market frictions, such works typically introduce ad hoc "wage rules" (e.g. Gornemann et al., 2021) or labor unions (e.g. Kekre, 2023) to avoid the bilateral bargaining problem. An exception is the recent work of Consolo and Hänsel (2023), who instead show that using an Alternative Offer Bargaining (AOB) protocol similar to Christiano et al. (2016) can avoid the wealth-dependence of the wage schedule. Additionally, there is also research that extends BHA models with labor market frictions using directed search, sidestepping the need for bargaining. Examples include Eeckhout and Sepahsalari (2023) or Chaumont and Shi (2022).

Finally, my work also relates to a technical literature extending the original EGM method to more complex settings. For example, Fella (2014) and Iskhakov et al. (2017) provide variants for models with both continuous and discrete household choices, while the

[1] While I do not address it in this paper, I conjecture that my method may also be useful for analyzing aggregate shocks using global methods à la Krusell and Smith (1998).

[2] In such cases, its computational benefits are probably even larger: If workers have a fixed lifetime of T periods, the KMS/BCK methods will have to search for T times as many wage functions in the outer loop, while iterating backwards T times using MIEGM provides them immediately.

methods of Hintermaier and Koeniger (2010) and Druedahl and Jørgensen (2017) allow for multiple assets. My MIEGM approach adds another extension for a different setting, which, importantly, can be combined with other EGM variants: This is particularly relevant for the discrete choice ones, as SaM models often feature discrete job separation/quitting decisions.

The remainder of the paper is structured as follows: Section 2 presents the numerical methods to be used, including MIEGM. Section 3 applies them to a stationary version of the KMS model, benchmarking MIEGM against previous algorithms. Section 4 considers an extension featuring endogenous separations as in Bils et al. (2011). Section 5 considers a model with intensive margin labor supply as in Nakajima (2012), which Section 6 then uses as example environment to demonstrate how my method can be used to study aggregate shocks and transition dynamics. Section 7 concludes.

2 Introducing the algorithms

After presenting an example environment and the existing methods to solve such models, this section introduces my *Match-Integrated Endogenous Grid Method* (MIEGM) for solving BHA-DMP models with bilateral bargaining.

2.1 The Environment

For simplicity, this section considers a stationary model. Time is discrete and runs forever. The model features workers that differ by current labor productivity s_i, which evolves stochastically according to some Markov process. A worker with current productivity s is assumed to be able to produce $y(s)$ units of output per period. Additionally, workers can be either employed (e) or unemployed (u) and are able to save in a risk-free asset a, which offers return r and is subject to an exogenous borrowing constraint $\underline{a} \leq 0$. An employed worker receives wage $w(s, a)$, which will be determined through Nash bargaining with their employer as specified below. An unemployed worker receives income $b(s)$, which may consists of home production and/or an UI benefit provided by the government.[3]

Employed workers get separated from their job at exogenous probability δ, while unemployed workers find work with probability p^{ue}. The latter usually depends on aggregate labor market tightness, determined in equilibrium through a free entry condition for firms. However, it is taken as given by worker and firm alike, so that its determination will be irrelevant for the bargaining outcome. The same holds for interest rate r.

[3] I abstract from dependence of unemployment benefit on wealth at this point. However, the Nakajima (2012) model studied in Section 5 features a wealth-dependent UI, which my method can handle as well.

Assuming a standard CRRA utility functions, the Bellman equations corresponding to the worker consumption-savings problem are thus

$$\begin{aligned} V^e(s,a) &= \max_{a'} \left\{ \frac{c^{1-\xi}-1}{1-\xi} + \beta \mathbb{E}\left[\delta V^u(s',a') + (1-\delta)V^e(s',a')\right] \right\} \\ &\text{s.t. } c = w(s,a) + (1+r)a - a' \end{aligned} \tag{1}$$

for employed workers and

$$\begin{aligned} V^u(s,a) &= \max_{a'} \left\{ \frac{c^{1-\xi}-1}{1-\xi} + \beta \mathbb{E}\left[p^{ue} V^e(s',a') + (1-p^{ue})V^u(s',a')\right] \right\} \\ &\text{s.t. } c = b(s) + (1+r)a - a' \end{aligned} \tag{2}$$

for unemployed workers.

The model also features firms which match with workers on the frictional labor market. Normalizing the output price to 1, the period profit of a firm matched with a worker of productivity s and wealth a is thus $y(s) - w(s,a)$. Under the common assumption that firms are risk-neutral and discount the future using the risk-free rate, the value function of such a firm is in turn defined by

$$J(s,a) = y(s) - w(s,a) + \frac{1}{1+r}(1-\delta)\mathbb{E}J(s',a'), \tag{3}$$

where a' is the optimal savings choice of a worker with wealth a'. The value of an unmatched firm is set to 0, as DMP-type models typically feature a free entry condition.

The wage $w(s,a)$ is determined through a Nash bargaining protocol in which the worker has bargaining power $\gamma \in (0,1)$, i.e. it will be given by

$$w(s,a) = \arg\max_{w} \left(V^e(w,s,a) - V^u(s,a)\right)^{\gamma} \left(J(w,s,a)\right)^{1-\gamma} \ . \tag{4}$$

From (4), it is clear why it is difficult solve such a model: (1), (2) and (3) all need to be solved numerically and depend on the wage schedule $w(s,a)$ in a complicated fashion, as the workers' savings decisions $a'(s,w)$ depends on the wage schedule, which in turn depends on the workers' savings decisions.

2.2 Solving the model: Previous approaches

As already indicated above, the previous approaches by KMS and BCK to solve such a model both use the idea of solving for the wage schedule as functional fixed point in an *outer loop*:

1. Start with a (discretized) candidate wage function $w^0(s,a)$ on a grid over (s,a).
2. Given the wage schedule, solve for value functions (1), (2) and (3) and worker policy functions by Value Function Iteration (VFI).
3. Get the wage schedule $\hat{w}(s,a)$ implied by the value functions.

(a) KMS get $\hat{w}(s,a)$ by solving (4) using numerical optimization.

(b) Other authors use the First Order Condition (FOC) of (4)

$$(1-\gamma)\frac{V^e(s,a)-V^u(s,a)}{c^e(s,a)^{-\xi}}+\gamma\frac{J(s,a)}{-1+\frac{1-\delta}{1+r}\mathbb{E}\frac{\partial J(s',a')}{\partial a'}\frac{\partial a'}{\partial w}}=0\ , \tag{5}$$

the denominator of the second term reflecting that the current wage affects the firm's continuation value through the workers' saving decision. Substituting (3), (5) can be rewritten as

$$\begin{aligned}\hat{w}(s,a)=y(s)&+\frac{1}{1+r}(1-\delta)\mathbb{E}J(s',a')\\&+\left(-1+\frac{1-\delta}{1+r}\mathbb{E}\frac{\partial J(s',a')}{\partial a'}\frac{\partial a'}{\partial w}\right)\frac{1-\gamma}{\gamma}\frac{V^e(s,a)-V^u(s,a)}{c^e(s,a)^{-\xi}}\end{aligned} \tag{6}$$

which provides a new candidate wage schedule for given value- and policy functions. To the best of my knowledge, this idea was first used by Bils, Chang, and Kim (2011), so I will henceforth refer to it as the BCK method. It is worth noting that several authors, including BCK, seem to use an incorrect version of FOC (5) that omits the derivative of the firm's continuation value. As I demonstrate in Appendix A, this yields an incorrect wage function differing from the numerical optimization approach.

4. Compare the implied wage schedule $\hat{w}(s,a)$ with the guess $w^0(s,a)$

   - If $\hat{w}(s,a)\approx w^0(s,a)$ for all (s,a) on the grid, stop.
   - If not, update the candidate wage schedule according to

$$w^0(s,a)=\zeta_w\hat{w}(s,a)+(1-\zeta_w)w^0(s,a)\ ,$$

   with $\zeta_w\in(0,1)$, and go back to step 2.

Essentially, the KMS and BCK algorithms differ only in how the wage schedule is updated: KMS solve the optimization problem defining the Nash solution (4) numerically, while BCK use the corresponding first order conditions.

2.3 Speeding things up: EGM

EGM has become a standard tool for solving BHA models numerically, combining high accuracy and speed due to avoiding numerical root-finding- or optimization procedures. It thus seems natural to also apply it to Step 2 of the solution procedures described in Section 2.2 above. In a BHA-DMP model, the savings choice of an unconstrained household will still be consistent with an Euler equation of the form

$$c^{\mathfrak{e}}(s,a)^{-\xi}=\beta\sum_{\mathfrak{e}'\in\mathfrak{E}}\sum_{s'\in\mathcal{S}}P^e(\mathfrak{e}'|\mathfrak{e})P^s(s'|s)V_a^{\mathfrak{e}'}(s',a')\ ,$$

where $P(e|u) = p^{ue}$ and $P^e(u|e) = \delta$. However, the marginal value functions $V_a^{\mathfrak{e}}$ fulfill

$$V_a^e(s,a) = \left(1 + r + \frac{\partial w(s,a)}{\partial a}\right) c^e(s,a)^{-\xi} \;\text{ and }\; V_a^u(s,a) = (1+r)\, c^u(s,a)^{-\xi}$$

by the Envelope Theorem: Compared to a standard consumption-savings model, V_a^e additional contains the derivative of the wage function, as additional assets do not only yield an interest rate but also affects future wage bargaining outcomes. As no Envelope Theorem is available for the latter, it has to numerically approximated by fitting an approximant/interpolant to the wage function and computing derivatives of the latter. Subsection 2.5 discusses some related issues that apply both to EGM and the MIEGM extension. Otherwise, for a given approximated $\frac{\partial w(s,a)}{\partial a}$ on the grid for (s,a), the algorithm proceeds as the well-known standard variant.

2.4 Solving for wages and value functions jointly: MIEGM

This section describes in detail how the Match-Integrated Endogenous Grid Method (MIEGM) can be used to solve for the value- and policy functions as well as the wage schedule jointly. The presentation corresponds to the stationary infinite horizon problem as described above. For the finite horizon case, instead of a previous guess, one would simply use the actual next period value- and policy functions, solved back from a final period with known terminal value. For ease of presentation, it is additionally assumed that individual labor productivity s follows a discrete Markov Chain with possible states $\mathcal{S} := \{s_1.s_2, \ldots, s_{ns}\}$ with transition probabilities $P^s(s'|s)$, consistent with the standard practice of discretizing continuously distributed income processes.
We start by fixing two ordered grids for household assets $\mathcal{A}_p := \{\underline{a}, a_1^p, \ldots, a_{n_p}^p\}$ and $\mathcal{A}_w := \{\underline{a}, a_1^w, \ldots, a_{n_w}^w\}$. On $\mathcal{A}_p$, we will represent the household- value and policy functions, while on $\mathcal{A}_w$, we will represent the wage schedule.[4] Additionally, we fix initial guesses for household value- and policy functions $V^{\mathfrak{e},0}(s,a)$, $a'^{\mathfrak{e},0}(s,a)$ for either employment status $\mathfrak{e} \in \mathfrak{E} := \{e,u\}$ as well as for the job value functions $J^0(s,a)$ and the wage schedule $w^0(s,a)$ The MIEGM algorithm then proceeds as follows:

1. **EGM step:** Given our current guess for the wage schedule $w^0(s,a)$, $a'^0(s,a)$, we first proceed as for a standard EGM and use

$$\hat{c}^{\mathfrak{e},1}(s;a') = \left(\beta \sum_{\mathfrak{e}'\in\mathfrak{E}} \sum_{s'\in\mathcal{S}} P^s(s'|s) V_a^{\mathfrak{e}',0}(s',a')\right)^{-\frac{1}{\xi}} \quad \forall\; a' \in \mathcal{A}_p\,, s \in \mathcal{S},\; \mathfrak{e} \in \mathfrak{E}$$

$$\text{with }\; V_a^{e,0}(s,a) = \left(1 + r + \frac{\partial w^0(s,a)}{\partial a}\right) c^{e,0}(s,a)^{-\xi} \;\text{ and }\; V_a^{u,0}(s,a) = (1+r)\, c^{u,0}(s,a)^{-\xi}$$

to get an update of the consumption- and saving policy functions on the endogenous *Cash-on-Hand* (CoH) grids

$$\hat{\mathcal{M}}^{\mathfrak{e}}(s) = \left\{\hat{c}^{\mathfrak{e},1}(s;\underline{a}) + \underline{a}, \ldots, \hat{c}^{\mathfrak{e},1}(s;a_{n_p}) + a_{n_p}\right\}$$

[4] I use the same asset grids for any worker productivity level s.

which contains the total amount of resources (income + asset wealth) that an household with income state $(s, \mathfrak{e})$ that chooses to save $a' \in \mathcal{A}_p$ must have had available. The derivative $\frac{\partial w^0(s,a)}{\partial a}$ has to be approximated numerically from the guesses $w^0(s, a)$ on the grid.

Notice that since we know the income of an unemployed worker to be $b(s)$, we can proceed as in the normal EGM for this group by backing out the an endogenous grid

$$\hat{\mathcal{A}}^u(s) := \left\{ m \in \hat{\mathcal{M}}^u(s) \ : \ (m - b(s))/(1 - r) \right\}$$

which contains the asset holdings an unemployed worker with productivity s must have had available if she chose to save the respective $a' \in \mathcal{A}_p$. Interpolating from $\hat{\mathcal{A}}^u(s)$ onto $\mathcal{A}_p$ then gives us the savings policy function for unemployed workers, so that we have updates $V^{u,1}(s, a)$, $a'^{u,1}(s, a)$.

However, for employed workers, this does not work, as we don't know the wage schedule $w^1(s, e)$ that would be needed to back out the respective endogenous grid for asset holdings. Nevertheless, the EGM step allows us to construct an interpolant of the employed worker savings' policy as function of their CoH, which we denote by $a'^e(s, m)$.[5]

2. **Obtaining wages:** From the previous step, we have obtained value- and policy functions for unemployed workers as well as an interpolant for the savings policy of employed workers as function of their CoH. This means that for a given candidate wage schedule $\hat{w}(s, a)$ on $\mathcal{A}_w$, we can evaluate the RHS of (6) and in turn use this equation back out the wage schedule $w^1(s, e)$ consistent with the Nash bargaining solution. While this can be done also in other ways, I solve the equation using a fixed point iteration approach:

   (a) Set the candidate wage schedule $\hat{w}(s, a) = w^0(s, a) \ \forall s \in \mathcal{S}, a \in \mathcal{A}_w$, so that the Cash-on-Hand of an employed worker on the grid will be $\hat{m}^e(s, a) = \hat{w}(s, a) + (1 + r)a$.

   (b) Using the interpolants from the previous step, we can obtain the corresponding value function as

$$\begin{aligned} \hat{V}^e(s, a) = & \frac{\left(\hat{m}^e(s, a) - a'^e\left(s, \hat{m}^e(s, a)\right)\right)^{1-\sigma} - 1}{1 - \sigma} \\ & + \beta \sum_{\mathfrak{e}' \in \mathfrak{E}} \sum_{s' \in \mathcal{S}} P^e(\mathfrak{e}'|e) P^s(s'|s) V^{\mathfrak{e},0}\left(s, a'^e\left(s, \hat{m}^e(s, a)\right)\right) \ . \end{aligned} \tag{7}$$

[5] As for the standard EGM, the borrowing constraint can be accounted for by using that any worker with less wealth/CoH than on the respective first grid point is constrained.

Use this value function to update the wage schedule through (6):

$$\hat{w}^1(s,a) = y(s) + \frac{1-\delta}{1+r}\sum_{s'\in\mathcal{S}} P^s(s'|s)J^0(s', a'^e(s, \hat{m}^e(s,a)))$$
$$+ \left(-1 + \frac{1-\delta}{1+r}\mathbb{E}\frac{\partial J^0(s', a'^e(s, \hat{m}^e(s,a)))}{\partial a'}\frac{\partial a'^e(s, \hat{m}^e(s,a))}{\partial m}\right)$$
$$\times \frac{1-\gamma}{\gamma}\frac{\hat{V}^e(s,a) - V^u(s,a)}{\left(\hat{m}^e(s,a) - a'^e\left(s, \hat{m}^e(s,a)\right)\right)^{-\xi}} \tag{8}$$

Again, the derivatives of J^0 and a'^e have to be approximated numerically.

(c) Compare the distance between the current candidate wage schedule $\hat{w}^1(s,a)$ and the previous guess $w^0(s,a)$ on $\mathcal{A}_w$. If it is small enough, then stop. If not, update the candidate wage schedule according to

$$\hat{w}^0(s,a) = \zeta_w\hat{w}^1(s,a) + (1-\zeta_w)\hat{w}^0(s,a) \quad ,$$

get the corresponding $\hat{m}^e(s,a)$ on $\mathcal{A}_w$ and go back to step (b).

MIEGM thus relies on the same updating equation as the BCK algorithm. As Maliar and Maliar (2014) noted, using fixed point iteration has the benefit of not requiring derivatives, which means that its computational costs do not increase substantially with the dimensionality of the system. A potential downside is that convergence cannot in general be guaranteed, although I did not face related problems: Practically, it was sufficient to adjust the updating parameter ζ_w for different models. Nevertheless, the use of fix point iteration is not necessary for MIEGM and other root finding methods could be considered instead. Note that either way, the procedure requires constructing interpolants for $a'^e(s,m)$, $V^{e,0}(s,a)$ and so on only once.

3. **Update the Rest:** Using the wage schedule $w^1(s,a)$ on grid $\mathcal{A}_w$, we can now update the value- and policy functions: Approximate w^1 on $\mathcal{A}_p$ and get the corresponding Cash-on-Hand $m^e(s,a)$. The latter can be used with the same interpolant $a'^e(s,m)$ to get the employed savings choices on the grid, which can then be used to get the update $V^e(s,a)$ as in (7) and the job value function according to

$$J^1(s,a) = y(s) - w^1(s,a) + \frac{1-\delta}{1+r}\sum_{s'\in\mathcal{S}} P^s(s'|s)J^0(s', a'^e(s', m^e(s,a))) \quad .$$

4. **Check for convergence:** Compare the distance between previous and updated value- and policy functions as well as the wage schedule on their respective grids. If they are small enough, then stop. If not, update the candidate value- and policy function according to

$$V^{e,0}(s,a) = V^{e,1}(s,a)$$
$$a'^{e,0}(s,a) = a'^{e,1}(s,a)$$
$$w^0(s,a) = w^1(s,a)$$
$$J^0(s,a) = J^1(s,a)$$

and go back to Step 1.

The algorithm above uses separate grids for the wage schedule and the household value- and policy functions. While this is not strictly necessary, it a) speeds things up by conducting the costly fixed point iteration at fewer points and b) it can be useful for accuracy by avoiding the accumulation of interpolation error.

Note that in its presented form, MIEGM is also applicable to compute perfect foresight transition dynamics, as Steps 1 to 3 combine the next period value functions with current interest- and job finding rates and obtain the current period values/policies. This forms the basis of the results in Section 6.

If one is only interested in solving a stationary infinite-horizon model, it can be possible to achieve additional computational speed by not conducting Step 2 every iteration. That is because for many models, J and w converge substantially faster than V. I will refer to this idea as AS-MIEGM (Accelerated Stationary MIEGM) below.

2.5 A note on Function Approximation

Compared to standard EGM, MIEGM requires approximating not only the levels but also derivatives of several unknown functions not subject to an Envelope Theorem: The job value function $J(s, a)$, the savings policy function $a'(s, m)$ as well as the wage schedule $w(s, a)$. It thus seems desirable to use approximation techniques providing for a smooth first derivative: For $J(s, a)$ and $a'(s, m)$, standard cubic spline interpolation worked well for all models I considered. However, approximating the wage function off-grid may require particular care: It is often characterized by a high-curvature region close to the borrowing constraint, in which standard cubic splines may end up featuring local non-concavities that can impede convergence for both VFI- and EGM-type algorithms.

A simple way to address this is (at the cost of some accuracy) to use a relatively wide-spaced $\mathcal{A}_w$ grid which tends to result in well-behaved interpolants. Alternatively, I found it useful to approximate the wage function off-grid by projecting the $w(s, a)$ on $\mathcal{A}_w$ on a set of Chebyshev polynomials of moderate order (typically 10-15 for every $s \in \mathcal{S}$), which provided better accuracy and stability for several models.[6]

3 Solving the stationary KMS model

This section applies the MIEGM algorithm to a stationary version of the KMS model, benchmarking it against the KMS and BCK algorithms. Prior to that, I briefly provide additional details on the model environment.

[6]Another remedy might be to use shape-constrained interpolation methods as e.g. discussed in Cai and Judd (2014).

Parameter	Description	Value
β	Worker discount factor	0.995
ξ	Worker risk aversion	1.0
$\underline{a}$	borrowing constraint	0.0
h	Home production	0.99
γ	Worker bargaining power	0.72
δ	Job separation rate	0.05
δ^k	Capital depreciation rate	0.01
α	Capital share	0.3
A_m	Matching function scale	0.6
κ	Vacancy posting cost	0.532
η	Matching function elasticity	0.72

Table 1: KMS calibration

3.1 The Environment

Regarding workers and firms, the KMS model is nested in the set-up described above: There is no skill heterogeneity, so that all workers have the same productivity $s = 1$. Additionally, firms produce using capital, so that their value function is

$$J(a) = \max_k \left\{ k^\alpha - (r + \delta^k)k - w(a) + \frac{1}{1+r}(1-\delta)\mathbb{E}J(a') \right\} \quad .$$

However, since their capital input can be freely adjusted, the respective first order condition implies simply implies $k = ((r+\delta)/\alpha)^{1/(\alpha-1)}$. The job finding probability p^{ue} is determined by a matching function of the Cobb-Douglas form $M = A_m U_t^\eta V_t^{1-\eta}$ so that $p^{UE} = A_m \theta^{1-\eta}$ where $\theta := V_t/U_t$ denotes labor market tightness. In general equilibrium, the latter is pinned down by a free entry condition of the form

$$\kappa = A_m \theta^{-\eta} \int J(a'^u(a)) \frac{f_u(a)}{u} da \tag{9}$$

where κ is a vacancy posting cost and $\frac{f_u(a)}{u}$ the density of unemployed workers over a.
I slightly deviate from the original KMS model by assuming that firms are owned by a set of risk-neutral entrepreneurs that discount the future at the current interest rate, and not by workers through share holdings. This simplifies the General Equilibrium asset market clearing condition, which, in turn, is given by

$$\int a f(a) da = (1-u)k \quad . \tag{10}$$

I adopt the benchmark KMS calibration, which is displayed in Table 1.

3.2 Numerical Details

For Bewley-type incomplete markets models, it is standard practice to use grids with more points close to the borrowing constraint, due to the more pronounced curvature of the policy functions in this region. In turn, grid $\mathcal{A}_w$ is specified to be exponentially spaced as

$$\mathcal{A}_i = \underline{a} + \exp(\mathbf{u}(\log(1 + a_{max}), n_i)) - 1$$

and $\mathcal{A}_p$ double-exponentially spaced as

$$\mathcal{A}_i = \underline{a} + \exp(\exp(\mathbf{u}(\log(1 + \log(1 + a_{max})), n_i)) - 1) - 1$$

where $\mathbf{u}(x_{max}, n_i)$ is a uniform grid on $[0, x_{max}]$ with n_i points. Choosing a more non-linear spacing for $\mathcal{A}_p$ typically improves accuracy. In the baseline application approximate all unknown functions using cubic splines except the wage function, which I fit using a 12th order Chebyshev polynomial (cf. Section 2.5). The grid sizes are $n_p = 500$ and $n_w = 75$. Appendix D additionally considers the results for the case in which w is interpolated using cubic splines.

All algorithms that are compared start with the same initial guesses. For the KMS algorithm, the 1-dimensional maximization problem as in (4) is solved using Golden Section Search, as in the codes for the original paper.[7] When necessary for general equilibrium, the joint distribution of workers over wealth and employment status is approximated using the non-stochastic simulation method by Young (2010).

I assess convergence of iterative procedures in terms of the sup norm: For all methods, I require 10^{-6} for the worker value functions, while for the firm value functions, I use 10^{-9}. The latter tend to converge much faster, so requiring higher accuracy is not very costly. The convergence criteria for the outer wage loop in the BCK/KMS methods is 10^{-4}, while for the MIEGM fixed point iteration, it is 10^{-10}.

The numerical performance of either method depends substantially on the updating parameter ζ_w: I use $\zeta_w = 0.08$ for the BCK method, $\zeta_w = 0.1$ for the KMS scheme and $\zeta_w = 0.75$ for MIGEM. As demonstrated in Appendix B, these value achieved particularly high computation speeds for the respective methods at KMS's calibration targets of $p^{ue} = 0.6$ and an interest rate of $r = 0.015$.

3.3 Comparison: Results

Before comparing the computational performance of the different methods, it is useful to ensure that the used methods indeed provide the same results: Firstly, it may not seem obvious that solving for the wage in an inner vs. an outer loop must yield the same outcome. Secondly, it is not generally clear that relying on an FOC should yield the same

[7] As of April 2025, these codes are available on Toshihiko Mukoyama's website under https://toshimukoyama.github.io/MyWebsite/programs.zip

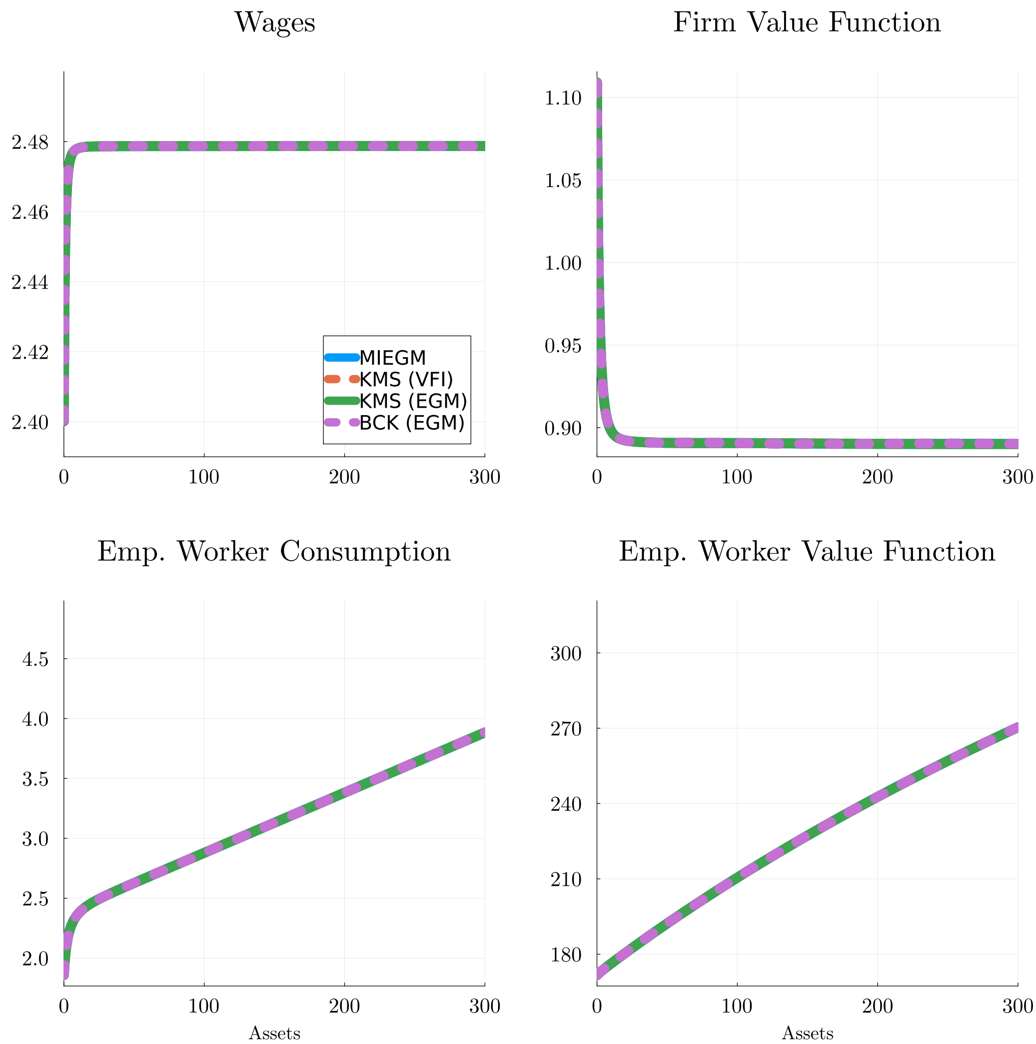

Figure 1: Results of the different solution methods for the KMS model

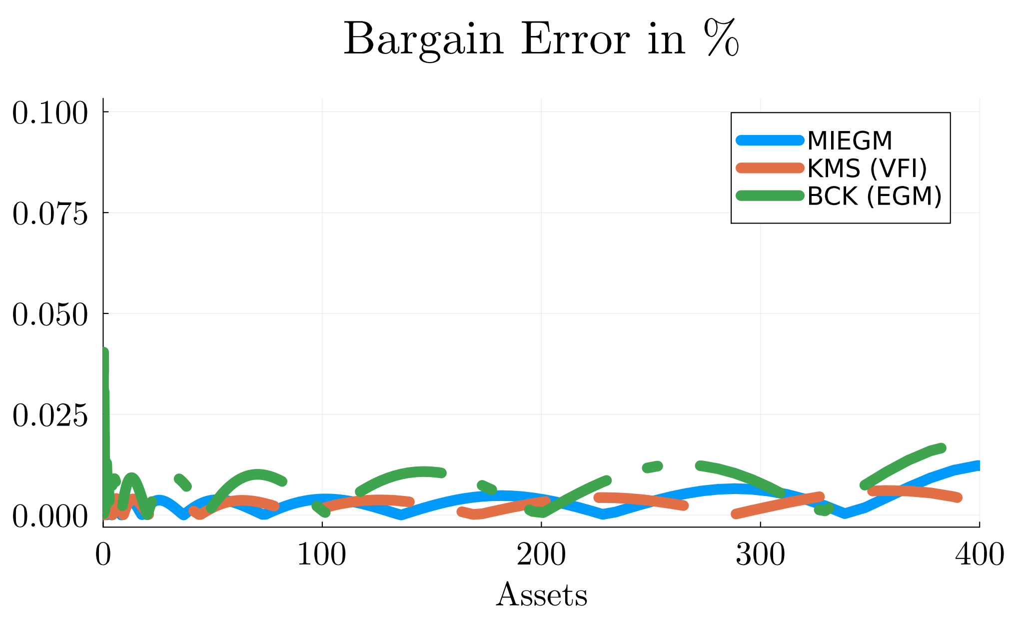

Figure 2: Wage accuracy check for different solution methods

result as numerically optimizing the Nash product (4), as it is typically not possible to prove the latter to be concave.

Figure 1 compares numerical results generated the different methods for the KMS model at said calibration targets. In particular, I compare MIEGM results with the ones generated by the original KMS method solving the household problem using VFI (not relying on the Euler equation) as well as the KMS- and BCK algorithms also using EGM. Reassuringly, the results are virtually indistinguishable for all methods, establishing the validity on relying on a joint solution approach and/or the Nash product FOC for backing out the wage function.

As stated above, a challenge for all these methods is the approximation of the wage functions. One might thus wonder if this can in fact be achieved well off the $\mathcal{A}_w$ grid. To check accuracy, one can obtain *Bargaining Errors* by computing both the LHS and RHS of (6) at other points and assess their relative difference. Note that such errors do not only reflect inaccuracies in the approximation of the wage function, but also for the Value- and Policy functions entering the RHS of (6). For the KMS model, I display such errors in Figure 2. As can be expected, they are somewhat higher in the high-curvature region of the wage function but overall very small, hardly exceeding 0.04%.

3.4 Comparison: Speed

Having established the validity of my MIEGM, I now assess whether the joint solution method provided by MIEGM can provide for computational speed-ups compared to the KMS and BCK algorithms using EGM. The respective results are displayed in Table 2: Its first row contains the computation times for solving the Worker- and Firm value func-

	AS-MIEGM	MIEGM	BCK (EGM)	KMS (EGM)	BCK	KMS
Partial eq.	0.558s	0.789s	1.205s	1.640s	14.38s	15.91s
GE Steady State	14.54s	17.16s	46.81s	38.83s	561.6s	375.0s

Note: Reported time is the best out of 3 runs. For additional details on the Steady State experiment, see text and Appendix C.

Table 2: Algorithm computing times

tions as well as the wage schedule in partial equilibrium, again for given $p^{ue} = 0.6$ and $r = 0.015$.[8] All algorithms relying on EGM are dramatically faster than the classic KMS and BCK-methods using VFI, which take at least $10\times$ as long to solve. Compared to the faster BCK-EGM method, moving to MIEGM further increases speed by 30-55%, depending on whether the standard or AS-MIEGM version is used. Hence, MIEGM seems clearly advantageous even though its relative efficiency gains are smaller compared to using EGM in the first place.

Additionally, things can look a bit different when one has to solve the model many times at different for different p^{ue} and r values, e.g. for finding a model's stationary equilibrium or calibration purposes. The second row of Table 2 displays the computation times for doing the former, using heuristic updating rules for p^{ue} and r that end up requiring solving the Partial Equilibrium problem almost 70 times to achieve convergence (additional details on this exercise are provided in Appendix C). As we see, using MIEGM is now more than 2 times faster than KMS and almost 3 times faster than the BCK algorithm. I interpret this as MIEGM benefitting more from better initial guesses becoming successively available, while BCK does so less.

4 Extension: Endogenous separations

Having demonstrated that MIEGM is as accurate and faster than the previous methods for solving BHA-DMP models with bilateral bargaining for a benchmark model, I now provide proof-of-concept of its applicability to richer environments with additional features. In turn, this section applies my MIEGM methods to a stationary version of the BCK model, which features jobs with heterogeneous productivity and endogenous separations and thus slightly deviates from the setting described in Section 2.

[8] All computations were conducted on a Dell XPS 15 notebook with Intel Core i7-11800H CPU, running Julia.

4.1 The Environment

In the BCK model, labor productivity is match- instead of worker-specific, so that all unemployed workers are identical. Additionally, there is disutility from working, so that unemployed workers receive a linearly additive utility benefit of B. These features imply that in the model, the joint surplus of low productivity matches can be negative, resulting in job separations. Due to Nash bargaining, a negative *joint surplus* means negative surplusses both for worker and firms, so both parties always agree on when to separate. Thus, it is not possible to distinguish between quits and layoffs. Finally, BCK assume that all new jobs have a common productivity level $s = s^e$, ensuring that all matches result in an employment relationship.

Practically, the above means that the Bellman equations corresponding to the worker consumption-savings problem are now

$$\begin{aligned} &V^{e,1}(s,a) = \max_{a'} \left\{ \frac{c^{1-\xi}-1}{1-\xi} + \beta \mathbb{E} V^{e,0}(s',a') \right\} \\ &\text{s.t. } c = w(s,a) + (1+r)a - a' \;\text{ and }\; V^{e,0}(s,a) = \max\left\{ V^{e,1}(s,a), V^u(a) \right\} \end{aligned} \tag{11}$$

for employed workers and

$$\begin{aligned} V^u(a) = \max_{a'} &\left\{ \frac{c^{1-\xi}-1}{1-\xi} + B + \beta \left[p^{ue} \mathbb{E} V^e(s^e,a') + (1-p^{ue}) \mathbb{E} V^u(a') \right] \right\} \\ &\text{s.t. } c = b + (1+r)a - a' \end{aligned} \tag{12}$$

for unemployed workers. For firms, it is

$$J^1(s,a) = s - w(s,a) + \frac{1}{1+r} \mathbb{E} J^0(s',a') \;\text{ with }\; J^0(s,a) = \max\left\{ J(^1 s,a), 0 \right\} \;\; .$$

Nothing fundamental changes about the wage bargaining compared to Section 2, except the value functions. Given that solving for the joint value- and policy functions does not depend on it, I will abstract from general equilibrium considerations and simply take r and p^{ue} as given below.[9] Of course, practically they would be determined by an outer "loop" searching for values so that an asset market clearing condition and a free entry condition for firms are satisfied. For the job productivity, I proceed as in the original paper and assume that it follows a discretized AR(1)-process with persistence $\rho_s = 0.97$ and standard deviation $\sigma_s = 0.13$, while the entry value is $s^e = 1$. Similarly, I follow BCK for the remaining parameters, which are summarized in Table 3.

4.2 Numerical Implementation

The model requires two modifications relative to the algorithm as described in 2: Firstly, after having updated wages and value functions, i.e. after step 3 in Section 2.4, one

[9] In fact, BCK always treat r as exogenous due to a "small open economy" assumption.

Parameter	Description	Value
β	Worker discount factor	0.995
ξ	Worker risk aversion	1.0
$\underline{a}$	borrowing constraint	-6.0
b	Home production	0.4
B	Value of leisure	0.15
γ	Worker bargaining power	0.5
r	SS interest rate	$1.06^{(1/12)} - 1$
p^{UE}	SS job finding rate	0.31

Table 3: BCk calibration

checks whether either the firm surplus $J(s,a)$ or worker surplus $V^e(s,a) - V^u(a)$ is negative for any $(s,a) \in \mathcal{S} \times \mathcal{A}_p$. If yes, save a "separation flag" and set $V^{e,0}(s,a) = V^u(a)$, $a'^e(s,a) = a'^u(a)$ and so on.

Secondly, the discrete separation choice can (potentially) introduce kinks in the household value function and discontinuous consumption/savings policies. I address this by replacing the standard EGM step as described in Section 2.4 with the version proposed by Fella (2014) that resorts to VFI in non-concave regions of the value function.[10]

Otherwise, I proceed similar as before, using (double) exponentially spaced grids between $\underline{a} = -6.0$ and $a_{max} = 500$ with $n_p = 600$ and $n_w = 125$ points. I approximate all unknown functions using cubic splines except the wage function, which I fit using a 15th-order Chebyshev polynomial between $\underline{a}$ and $a = 200.0$ and by cubic spline interpolation beyond that. The wage updating parameter is $\zeta_w = 0.5$.

4.3 Results

Figure 3 displays various model results for the stationary BCK model, with different colors indicating workers with different s values. Workers that decide to separate from their job are assigned the respective unemployment value- and policy function as well as income $b = 0.4$. This explains the kinks in the upper-left panel. Overall, many patterns are reminiscent of the KMS model: The wage schedule is increasing in individual assets but becomes rather flat once a worker is sufficiently wealthy. In fact, the wealth dependence seems to more pronounced and the slope of the wage functions remain positive even at high wealth levels due to the additive disutility of working. We also see that the model features low-productivity jobs that workers quit only if wealthy enough.

Again, it is useful to consider the accuracy of the solution by computing bargaining errors: These are displayed in Figure 4 for the different job productivity levels. They are

[10] Other approaches to dealing with the discrete choice as proposed e.g. by Iskhakov et al. (2017) should be applicable as well. Smoothing out the discrete choice by taste- or productivity shocks may however require additional modification, as their realization could affect the bargaining outcome.

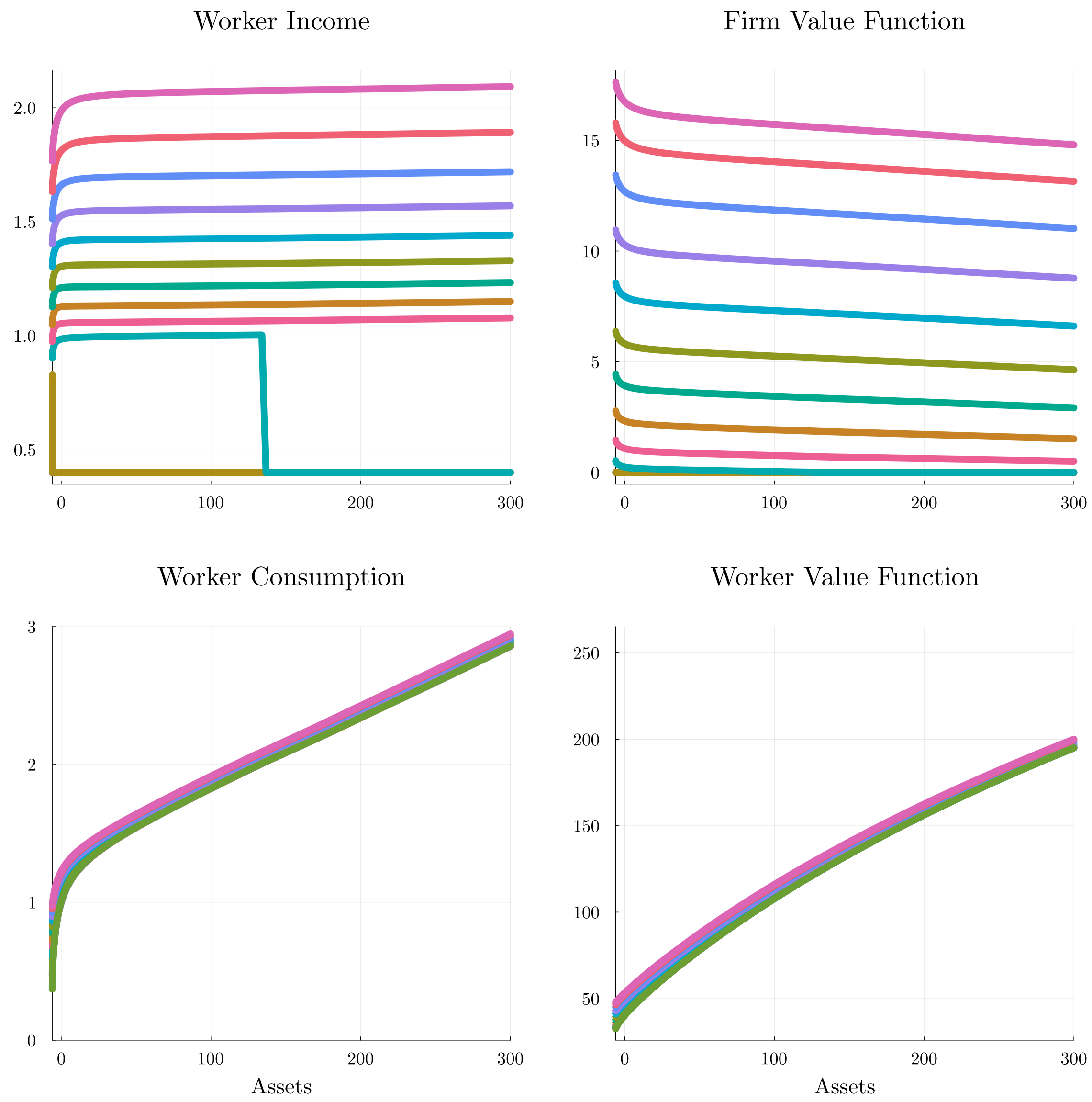

Figure 3: Results BCK setup

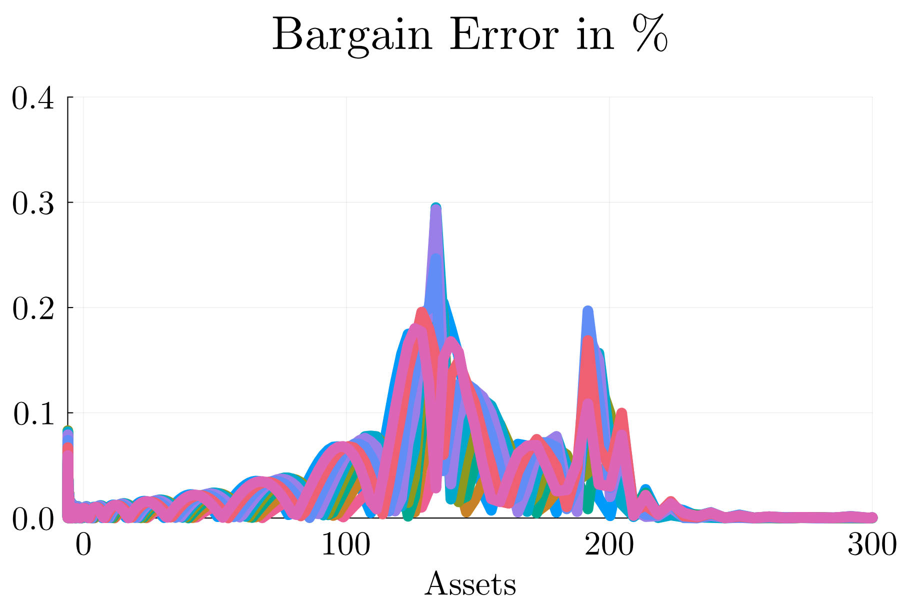

Figure 4: Wage accuracy check for BCK model

somewhat higher compared to the KMS model, peaking at almost 0.3% in the area close to the quit cut-off level for one of the worker types. But overall, the magnitude of the errors appears satisfactory.

5 Extension: Intensive margin labor supply

In various contexts, it may be relevant to also model intensive margin labor supply in addition to DMP labor market frictions. As I demonstrate, MIEGM can also handle this case: In particularly, I apply it to a model with Greenwood et al. (1988) (GHH) preferences as studied by Nakajima (2012). This choice reflects two considerations: Firstly, as will become clear below, models with GHH preferences are somewhat less obvious to solve using (MI)EGM, as one cannot even back out a households' endogenous *Cash-on-Hand* grid from the inverted Euler equation in this case. Secondly, Nakajima (2012) found GHH preferences helpful to generate realistic unemployment fluctuations, a well known challenge for the standard DMP model (cf. Shimer, 2005). This makes the model more interesting for demonstrating the usefulness of MIEGM for studying aggregate shocks.

5.1 The Environment

The model extends on the set-up from Section 2 in the following way: Workers do not only match with firms on a frictional labor market à la DMP, but also face an endogenous labor supply decision while on the job. For the latter, they have to trade off additional income with the discomfort of working more. In particular, a workers' period utility over consumption c_{it} working time l is given by

$$u(c_{it}, l_{it}) = \frac{\left(c_{it} - \psi \frac{l_{it}^{1+\frac{1}{\eta}}}{1+\frac{1}{\eta}}\right)^{1-\xi} - 1}{1-\xi}$$

A worker with productivity s is assumed to supply sl_{it} efficiency units of labor by working l_{it} hours. I follow Nakajima (2012) by defining the wage as *share* of the match output going to the worker, i.e., if the match can produce p units of output per efficiency units of labor, the worker will receive an income of $(1-\tau)pslw(s,a)$, where $(1-\tau)$ is a labor income tax. For unemployed workers, $l_{it} = 0$ and income will be given by an exogenously fixed UI benefit $b(s,a)$ that will be calibrated to provide a constant replacement rate χ in steady state, i.e.

$$b(s,a) = \chi(1-\tau)sp_{ss}l_{ss}(s,a)w_{ss}(s,a) \ .$$

The recursive representation of a workers dynamic problem is then given by

$$\begin{aligned} V^e(s,a,l) &= \max_{a'} \left\{ u(c,l) + \beta \mathbb{E}\left[\delta(1-p^{ue})V^u(s',a',l') + (1-\delta(1-p^{ue}))V^e(s',a',l')\right] \right\} \\ \text{s.t. } & c = (1-\tau)psw(s,a) + T + (1+r)a - a' \end{aligned} \tag{13}$$

for employed workers and

$$\begin{aligned} V^u(s,a) &= \max_{a'} \left\{ u(c,0) + \beta \mathbb{E}\left[p^{ue}V^e(s',a',l') + (1-p^{ue})V^u(s',a',0)\right] \right\} \\ \text{s.t. } & c = b(s,a) + T + (1+r)a - a' \end{aligned} \tag{14}$$

for unemployed workers. T denotes a lump-sum transfer/tax, which, in the Nakajima (2012) model consists of firm profits and the government's revenue surplus/deficit. Note that workers are allowed to immediately search for a new job after losing their old one, so the probability to become unemployed is $\delta(1-p^{ue})$ instead of δ as in the previous models.

A well known feature of GHH preferences is that it does not feature any wealth effects on labor supply, so that a workers' decision on how much to work depends solely on the current wage rate. In particular, for the specification considered here, optimal labor supply will be given by

$$l(s,a) = \left(\frac{(1-\tau)psw(s,a)}{\psi}\right)^{\eta} \ . \tag{15}$$

On the firms' side, the value function is given by

$$J(s,a,l) = psl(1-w(s,a)) + \frac{1-\delta}{1+r}\mathbb{E}J(s',a')$$

As in KMS, Nakajima (2012) in fact assumes firms to operate with a Cobb-Douglas production technology $ZK^{\alpha}L^{1-\alpha}$, with firms being able to flexibly adjust their capital stock. In turn, due to constant returns to scale, every firm will choose the same capital-labor ratio k^* implied by the first order condition $\alpha Z(k^*)^{\alpha-1} = r + \delta^k$. Hence, $p = Z(k^*)^{\alpha} - (r+\delta^k)k^*$.

In principle, bargaining is now both about the wage w and hours worked l. However, given that the latter is completely determined by the former, the match can equivalently bargain over the wage rate w only, with the worker supplying any amount of hours she likes afterwards. Thus, the bargained wage rate will solve

$$w(s,a) = \arg\max_{w} \ \left(V^e(w,s,a,l(s,w)) - V^u(s,a)\right)^{\gamma} \left(J(w,s,a,l(s,w))\right)^{1-\gamma} \quad , \tag{16}$$

the F.O.C.'s of which now imply

$$\gamma \frac{J(s,a,l)}{-psl + \frac{\partial l}{\partial w} ps(1-w) + \frac{1-\delta}{1+r}\mathbb{E}\frac{\partial J}{\partial a'}\frac{\partial a'}{\partial w}} + (1-\gamma)\frac{V^e(s,a,l) - V^u(s,a)}{(1-\tau)psl\frac{\partial u(c,l)}{\partial c}} = 0 \ .$$

In contrast to the models above, the LHS of this equation now takes into account that the firms' surplus changes in the wage rate also through its effect on labor supply. Finally, the model is characterized by the following equilibrium conditions:

- An asset market clearing condition. Denoting by $k(s,a)$ the amount of capital demanded by a firm matched with an (s,a)-type worker, this condition is given by

  $$\int\int k(s,a) f(e,s,a) ds da = \sum_{\mathfrak{e}\in\mathfrak{E}} \int\int a f(\mathfrak{e},s,a) ds da$$

  where $f(e,s,a)$ denotes current the distribution of workers over their idiosyncratic employment state, productivity, and asset holdings. Given that all firms will choose the same capital-labor ratio $k^* = \left(\frac{r+\delta}{\alpha}\right)^{1/(\alpha-1)}$ given the realized interest rate, on can equivalently write this condition as

  $$k^* L = A \quad \text{with} \quad L := \int sl(s,a) df^f(s,a) \ \text{ and } \ A := \sum_{\mathfrak{e}\in\mathfrak{E}} \int\int a f(\mathfrak{e},s,a) ds da. \tag{17}$$

  A represents total assets held by households and L the aggregate amount of efficiency units of labor supplied.

- A free entry condition for firms. Given vacancy cost κ, this would be given by

  $$\kappa = p^{vf}(\theta)\frac{J^S}{S} \tag{18}$$
  $$\text{with } \ J^S := \delta^s \int\int J(s,a) f(e,s,a) ds da + \int\int J(s,a) f(u,s,a) ds da$$
  $$\text{with } \ S := \delta^s \int\int f(e,s,a) ds da + \int\int f(u,s,a) ds da$$

  S is the mass of all workers searching for jobs in a given period (unemployed and newly separated), while J^S is the total surplus firms could generate by matching with these workers.[11] Since matching is random, the expected value of meeting a worker is thus J^s/S. p^{vf} is the vacancy filling probability depending on the labor market tightness $\theta := V/S$ through a matching function $M(S,V) = A_m S^{\alpha_m} V^{1-\alpha_m}$.

[11] I consider J^S and S as separate model variables, as they are both linear in the distribution, while the expected value of meeting a worker $\frac{J^S}{S}$ is not. Avoiding variables that depend non-linearly on the distribution simplifies the implementation of the Auclert et al. (2021) method.

- An equilibrium condition determining the transfer, which is given by firm profits net of vacancy positing cost as well as the government budget constraint. As the latter only raises taxes to finance the transfer, it is given by

$$T = \pi + gs - \kappa V \tag{19}$$
$$\text{with } \pi := \int\int p(s,a)sl(s,a)(1-w(s,a))f(e,s,a)dsda$$
$$\text{and } gs := \tau\int\int p(s,a)sl(s,a)w(s,a)f(e,s,a)dsda - \int\int b(s,a)f(u,s,a)dsda$$

  π denote the firm profits (output net of wage payments) and gs the government surplus/deficit (tax revenue minus UI payments).

- A goods market clearing condition will additionally hold due to Walras' law.

The calibration of the model is summarized in Table 4. Most parameters are taken from the original paper, but I re-calibrate β, ψ as well as κ and A_m to match the paper's targets of an aggregate capital-output ratio of 10, average hours worked of 1/3 among employed workers and an unemployment rate of 5.67% at $\theta = 1$ in steady state.[12] For the individual skill process, I follow Nakajima (2012) and assume it to follow a discretized AR(1)-process with persistence $\rho_s = 0.9956$ and standard deviation $\sigma_s = 0.0323$ which I discretize into 9 states using the Rouwenhorst (1995) method.

5.2 Adapting MIEGM to GHH preferences

Using MIEGM with GHH preferences as in the model requires some changes compared to the algorithm description in 2.4. Firstly, as the consumption and labor and non-separable in the households utility, inverting the Euler equation for next period guesses of $c(s,a)$ and $l(s,a)$ only yields

$$\hat{c}^{\mathfrak{e},1}(s;a') - \psi\frac{\left(\hat{l}^{\mathfrak{e},1}(s;a')\right)^{1+\frac{1}{\eta}}}{1+\frac{1}{\eta}} = \left(\beta\sum_{\mathfrak{e}'\in\mathfrak{E}}\sum_{s'\in\mathcal{S}}P^s(s'|s)V_a^{\mathfrak{e}',0}(s',a',l')\right)^{-\frac{1}{\xi}} \quad \forall\ a'\in\mathcal{A}_p, s\in\mathcal{S},\ \mathfrak{e}\in\mathfrak{E} \tag{20}$$

$$\text{where } V_a^{e,0}(s,a,l) = \left(1+r+(1-\tau)psl(s,a)\frac{\partial w(s,a)}{\partial a}\right)\left(c-\psi\frac{l(s,a)^{1+\frac{1}{\eta}}}{1+\frac{1}{\eta}}\right)^{-\xi}$$

i.e. a linear combination of the households consumption and the labor disutility term. This means the backward step does not provide us with consumption policy function on an endogenous Cash-on-Hand grid as before, but only tells us what that linear combination would be for an household with $s\in\mathcal{S}$ that decides to save $a'\in\mathcal{A}_p$.[13] Notice, though,

[12] I was not able to match the paper's targets at its stated calibration, which I suspect to be due to some additional normalization(s) not explicitly reported.

[13] Note that for unemployed households, this problem does not arise, as they are known to supply $l = 0$, so that c and an endogenous grid for m/a can be backed out.

Parameter	Description	Value
β	Worker discount factor	0.9844
ξ	Worker risk aversion	1.5
ψ	Labor Disutility parameter	6.16
η	Labor supply elasticity	0.5
$\underline{a}$	borrowing constraint	0.0
χ	UI replacement rate	0.64
γ	Worker bargaining power	0.0701
δ	Job separation rate	0.1
α	capital share	0.289
α_m	Matching elasticity	0.66
A_m	Matching efficiency	0.6245
κ	Vacancy posting cost	0.079
δ^k	Capital depreciation rate	0.015
τ	Tax rate	0.0370
Z_{ss}	Steady State TFP	0.51
r	SS interest rate	0.0139

Table 4: Calibration of Nakajima (2012) model

that given labor supply schedule (15), we have that for a worker's Cash-on-Hand $m(s,a)$

$$
\begin{aligned}
m(s,a) - \frac{l^{1+\frac{1}{\eta}}}{1+\frac{1}{\eta}} &= (1+r)a + T_t + (1-\tau)psw(s,a)\left(\frac{(1-\tau)psw(s,a)}{\psi}\right)^{\eta} - \frac{\left(\frac{(1-\tau)psw(s,a)}{\psi}\right)^{\eta+1}}{1+\frac{1}{\eta}} \\
&= (1+r)a + T_t + \frac{1}{1+\eta}(1-\tau)sl(s,a)w(s,a) := \tilde{m}(s,a) \ .
\end{aligned}
$$

In words, the value one obtains through subtracting the labor disutility term from the definition of Cash-on-Hand (CoH) equals an *adjusted* CoH $\tilde{m}(s,a)$ with labor income rescaled by $1/(\eta+1)$. Moreover, we can obtain an endogenous grid for this $\tilde{m}(s,a)$ by simply adding a' to the LHS of (20). As $\hat{\tilde{m}}$ can always be constructed for a given candidate wage schedule $\hat{w}(s,a)$, I set up the interpolators for worker value- and policy functions in terms of this variable instead of $m(s,a)$. While this admittedly lacks the theoretical foundation of treating policies as function of regular $m(s,a)$, this approach turned out to work well in practice.[14] Afterwards, I proceed as outlined in 2.4 with the additional difference that

[14]Of course, the accuracy of the solution can be verified ex-post by computing Euler equation errors.

the updating equation for the fix point iteration is now given by

$$\hat{w}^1(s,a) = \frac{1}{psl(s,\hat{w}^0(s,a))}\Bigg(psl(s,\hat{w}^0(s,a)) + \frac{1-\delta}{1+r}\mathbb{E}J(s',a'^{,e}(s,\hat{\tilde{m}}^e(s,a)))$$
$$+\frac{1-\gamma}{\gamma}\frac{-psl(s,\hat{w}^0(s,a)) + \frac{\partial l}{\partial w}ps(1-\hat{w}^0(s,a)) + \frac{1-\delta}{1+r}\mathbb{E}\frac{\partial J}{\partial a'}\frac{\partial a'^e}{\partial \hat{\tilde{m}}^e(s,a)}\frac{\partial \hat{\tilde{m}}^e(s,a)}{\partial w}}{(1-\tau)psl(s,\hat{w}^0(s,a))(\hat{\tilde{m}}^e(s,a)-a'^e(s,\hat{\tilde{m}}^e(s,a)))^{-\xi}}\left(\hat{V}^e(s,a,l)-V^u(s,a)\right)\Bigg)$$

$$\text{where } \frac{\partial l}{\partial w} = \frac{\eta}{\hat{w}^0(s,a)}\left(\frac{(1-\tau)ps\hat{w}^0(s,a)}{\psi}\right)^{\eta} \text{ and } \frac{\partial \hat{\tilde{m}}^e(s,a)}{\partial w} = \frac{1-\tau}{1+\eta}ps(\hat{w}^0(s,a)\frac{\partial l}{\partial w} + l(s,\hat{w}^0(s,a)))\ .$$

This equivalent to (8) used in Section 2 is now slightly more tedious due to the additional terms introduced by the endogenous labor supply. Importantly, though, implementing it does not pose any additional conceptual challenges, and it is still the case that one needs to construct interpolators only once for conducting the fix-point iteration.

5.3 Numerical Implementation

An additional complication of the model presented above is the UI schedule in the Nakajima (2012) model: While a wealth-dependent UI schedule by itself poses no conceptual problem for my algorithm, the fact that it depends explicitly on the entire wage schedule due to the replacement rate assumptions make things complicated: The wage schedule depends on the UI schedule but also vice versa. While I explored ways to solve for both in the inner MIEGM loop, such procedures failed to converge for various specifications. Thus, for my baseline implementation I update the UI schedule in the outer calibration loop.

Besides the changes in the algorithm described in the previous section, I use similar specifications as before. I generate my grids with $a_{max} = 300.0$, $n_p = 1000$ and $n_w = 125$, but now use double exponentially spaced grids for both $\mathcal{A}_w$ and $\mathcal{A}_p$ and resort to cubic splines for approximating the wage functions. These choices resulted in somewhat higher accuracy. To ensure stability of the inner fixed point iteration given the low value $\gamma = 0.0701$, I use a relatively low updating parameter $\zeta_w = 0.1$.

5.4 Results

Figure 5 displays various model results in the stationary steady state for workers with relatively high labor productivity (blue line), workers with median productivity (red line) and workers with low productivity (black line). Firstly, the wage schedule is increasing in wealth, as in the previous models. However, the gradient along the wealth dimension is very small except for low productivity workers very close to the borrowing constraint. Overall, this does not seem surprising, given the properties of GHH preferences that eliminate wealth effects on labor supply. As in Nakajima (2012), I find that high productivity workers end up receiving a slightly lower wage rate (as a share of match

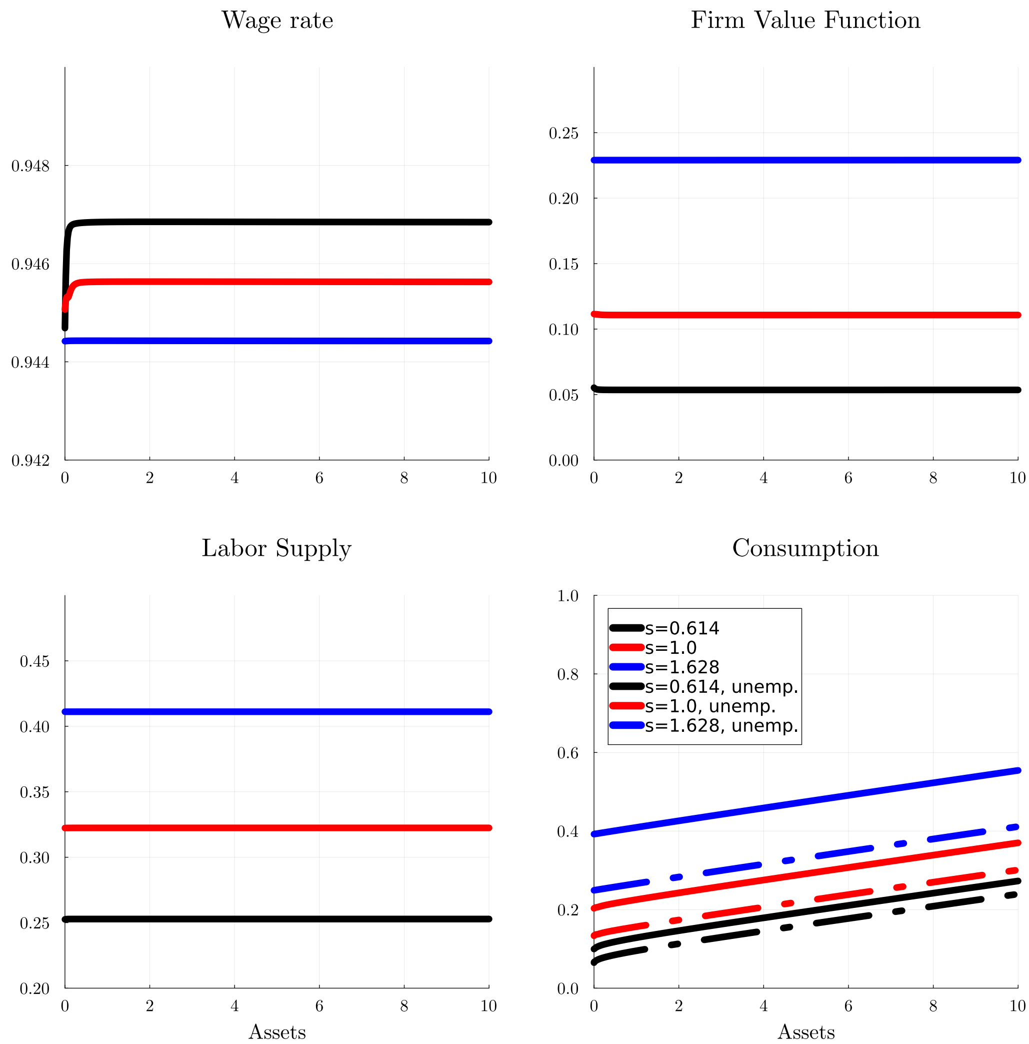

Figure 5: Model Results at Steay State

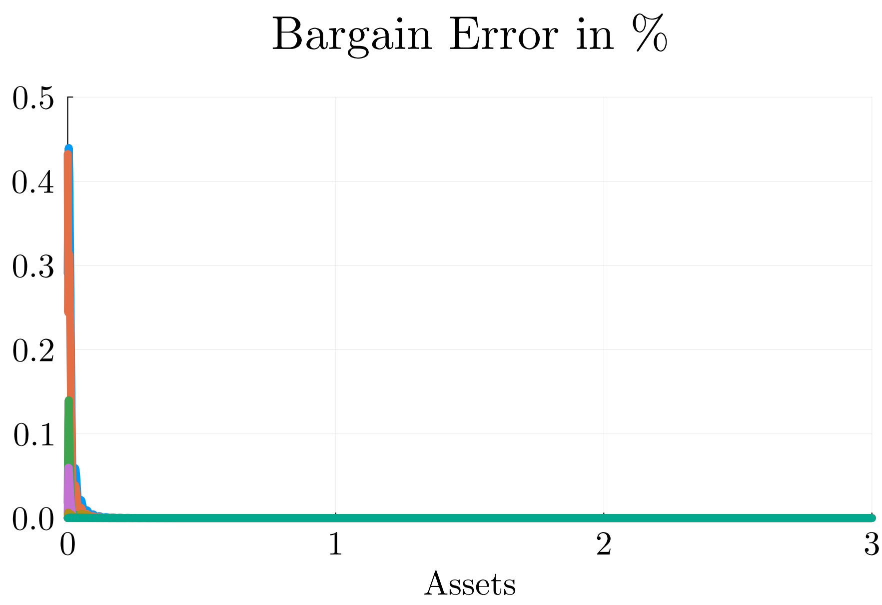

Figure 6: Wage accuracy check for Nakajima (2012)

output) than low productivity workers. In the bottom-left panel, we also see a large labor supply gradient between workers with different s: Workers with high productivity tend to work substantially more hours than low productivity workers. This somewhat unrealistic feature could presumably be mitigated if disutility from working were to also depend on s. Finally, the bottom-right panel shows the consumption choices of workers in the different skill groups, both if working (solid line) and unemployed (dashed line). The consumption profiles are fairly linear along the wealth dimension, reflecting the relatively low amount of idiosyncratic risk present in the model calibration. However, as already noted by Nakajima (2012), we see substantial differences between the consumption of employed and unemployed workers even for very rich workers. This result, due to the complementarity between labor supply and consumption introduced by the GHH preferences, raises the question to what extent empirically observed consumption effects of income changes due to unemployment might be attributable to preferences compared to other factors such as borrowing constraints.

We should again consider the accuracy in terms of Bargaining Errors: Figure 6 displays them for the different job productivity levels. Unsurprisingly, errors in the flat parts of the wage schedule are very low but we get some errors up to 0.4-0.5% for low productivity workers close to the borrowing constraint. Overall, though, the average error (weighted by the steady state distribution) is only approx. 0.01%, so accuracy seems satisfactory overall.

6 Aggregate Shocks

The previous sections have used MIEGM to solve stationary incomplete markets models with bilateral wage bargaining. However, many research- and policy questions also require the analysis of aggregate shocks or transition dynamics following policy changes. I demonstrate here that my novel method can also be useful for such purposes: In particular, I explain that the joint formulation of the worker- and firm problems used by MIEGM makes it possible to apply the "fake news" algorithm developed by Auclert et al. (2021) to obtain the Sequence Space Jacobian of their joint "Block", which can in turn be used to obtain linearized model impulse responses or to compute non-linear transition dynamics using quasi-Newton methods.[15] Thus, at least the second part of Auclert et al. (2021)'s assertion that their method is not applicable to "*labor-search models with wage posting or individual bargaining*" (p.2391) is in fact incorrect.

6.1 Why does it work?

Auclert et al. (2021) demonstrate that their method is applicable to environments of the form

$$\mathbf{v_t} = v(\mathbf{v}_{t+1}, \mathbf{X}_t) \tag{21}$$

$$\mathbf{D}_{t+1} = D(\mathbf{v_{t+1}}, \mathbf{X}_t, \mathbf{D}_t) \tag{22}$$

$$Y_t = y(\mathbf{v}_{t+1}, \mathbf{X}_t, \mathbf{D}_t) \tag{23}$$

The first line concerns the value- and policy functions of the model's heterogeneous agents (collected in $\mathbf{v}_t$), which are allowed to only depend recursively on their future values $\mathbf{v}_{t+1}$ as well as an vector of aggregate variables $\mathbf{X}_t$. The second line concerns the law of motion of the distribution $\mathbf{D}$, while the final line concerns aggregate outcomes $\mathbf{Y}_t$. An important restriction imposed by this form is that the idiosyncratic choices and values of the heterogeneous agents can not depend on the distribution but only a few aggregate variables. For BHA-DMP models with bilateral bargaining, this may at first seem to not be the case, as the wage schedule is given by an entire function. However, once we include both worker- and firm value- and policy functions into $\mathbf{v} := [V_t, J_t]$, the determination of the wage function becomes part of $v(\cdot)$.

It is important to notice that all that is needed for determining the worker- and firm value functions (as the well as the wage schedules) are the time paths of a few aggregate variables: In the case of the Nakajima (2012) model, these are the interest rate r, the job finding rate p^{UE}, the lump sum transfer T as well as aggregate shocks (e.g. to TFP) that other-

[15]The Joint Formulation also fits the model structure required by the methods recently proposed by Bhandari et al. (2023) and Boehl (2023), which facilitate solving for second-order perturbation solutions and non-linear responses following large unanticipated "MIT" shocks, respectively. It should thus also be possible to use these methods for analyzing BHA business cycle models with bilateral wage bargaining.

wise affect the productivity of a given match. In particular, they do not depend on the aggregate distribution **D** given these variables.[16] Thus, the joint recursive formulation of the worker- and firm problem fits into the form required by Auclert et al. (2021).

6.2 The Nakajima (2012) model as example environment

As stated above, I will use the Nakajima (2012) model as example environment to demonstrate the usefulness of MIEGM for analyzing aggregate shocks. To compute linearized impulse responses to some aggregate shock Z around the stationary steady state of a model, Auclert et al. (2021) propose to first compute so-called Sequence Space Jacobians (SSJs) with respect to the shock as well as a small set of aggregate variables from which the remaining equilibrium objects can be recovered. These Jacobians can then be used to compute aggregate impulse responses by relying on the Implicit Function Theorem.

For that purpose, these authors propose to split models into "blocks" ordered along a Directed Acyclical Graph (DAG), along which the Jacobians are accumulated using the chain rule. Additionally, they propose a fast "fake news" algorithm to compute the Jacobian of the heterogeneous agent block, the most complicated object of the model which combines (21) and (22).

Figure 7 shows an example DAG for the Nakajima (2012) model, with three "unknowns" r_t, θ_t, T_t, which are mapped along the DAG to the equilibrium targets, which correspond to equilibrium conditions (19), (18) and (17). The "Heterogeneous Agent Block" is presented as "Joint Block" combining the recursive problem of workers & firms with the distribution, the Sequence Space Jacobian of which can be obtained using the mentioned "fake news" algorithm. I decided to specify match productivity p_t as defined above as an explicit input into the Joint Block, but alternatively, one could also dispense with the "Firm Choice" block and determine p_t from r_t and Z_t within the "Joint Block".[17]

6.3 Obtaining the Joint Sequence Space Jacobian

To obtain the SSJs of what they call "Heterogeneous Agent" blocks, Auclert et al. (2021) use the responses to a single perfect foresight perturbation of the block inputs at $\mathcal{T}$ periods in the future from the steady state. As they describe in their paper, the resulting policy and value functions can then be used to compute the Jacobian of the block with respect to the input sequences from time 0 (steady state) to $\mathcal{T}$.

As outlined above, for the Nakajima (2012) model the inputs to the Joint Block are the

[16] This may not be the case for some models featuring on-the-job search. However, I conjecture that it should still be possible to fit many such models into the form (21)-(23) by including a finite number of distributional moments into X.

[17] The "Firm choice" block maps interest rate r_t and TFP Z_t to the optimal capital-labor ratio choice $k^* = \left(\frac{r_t+\delta^k}{\alpha Z_t}\right)$ and $p_t = Z_t(k_t^*)^\alpha - (r+\delta^k)k_t^*$.

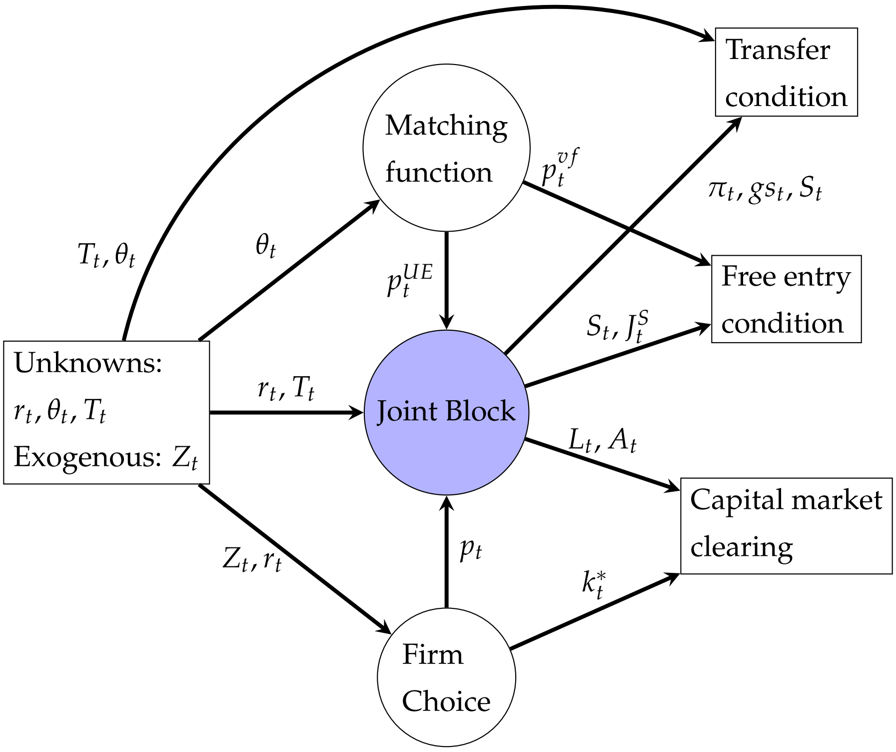

Figure 7: Example DAG for the Nakajima (2012) model

interest rate r_t, the job finding rate p_t^{UE}, the lump sum transfer T_t as well as match productivity p. We thus need to compute $V(\mathfrak{e}, s, a)$, $a'(\mathfrak{e}, s, a)$, $l(s, a)$, $w(s, a)$ as well as $J(s, a)$ along the perfect foresight paths after either of them changed by a small amount ϵ at time $\mathcal{T}$.[18]

Using MIEGM, this is straightforward: Given a perturbed sequence of block inputs, one gets the value- and policy functions by iterating backwards from $\mathcal{T} + 1$ using MIEGM. This does not require any modifications of the algorithms described previously except perhaps including the "re-centering"-step Auclert et al. (2021) recommend to tackle steady state inaccuracies. Since the model does only feature one distribution (the distribution of workers also captures the distribution of matched firms), one can follow Auclert et al. (2021) for the respective part of the "fake news" algorithm. Notice that if employed workers can immediately search again if losing their job, the distribution at any point t becomes endogenous to contemporaneous "shocks" in input p_t^{ue}, requiring the corresponding version of the algorithm to be used.

The reader is referred to Appendix E and the original Auclert et al. (2021) paper for implementation details.

[18]To ensure accuracy, I conduct these perturbations using the `ForwardDiff.jl` automatic differentiation routine (Revels et al., 2016).

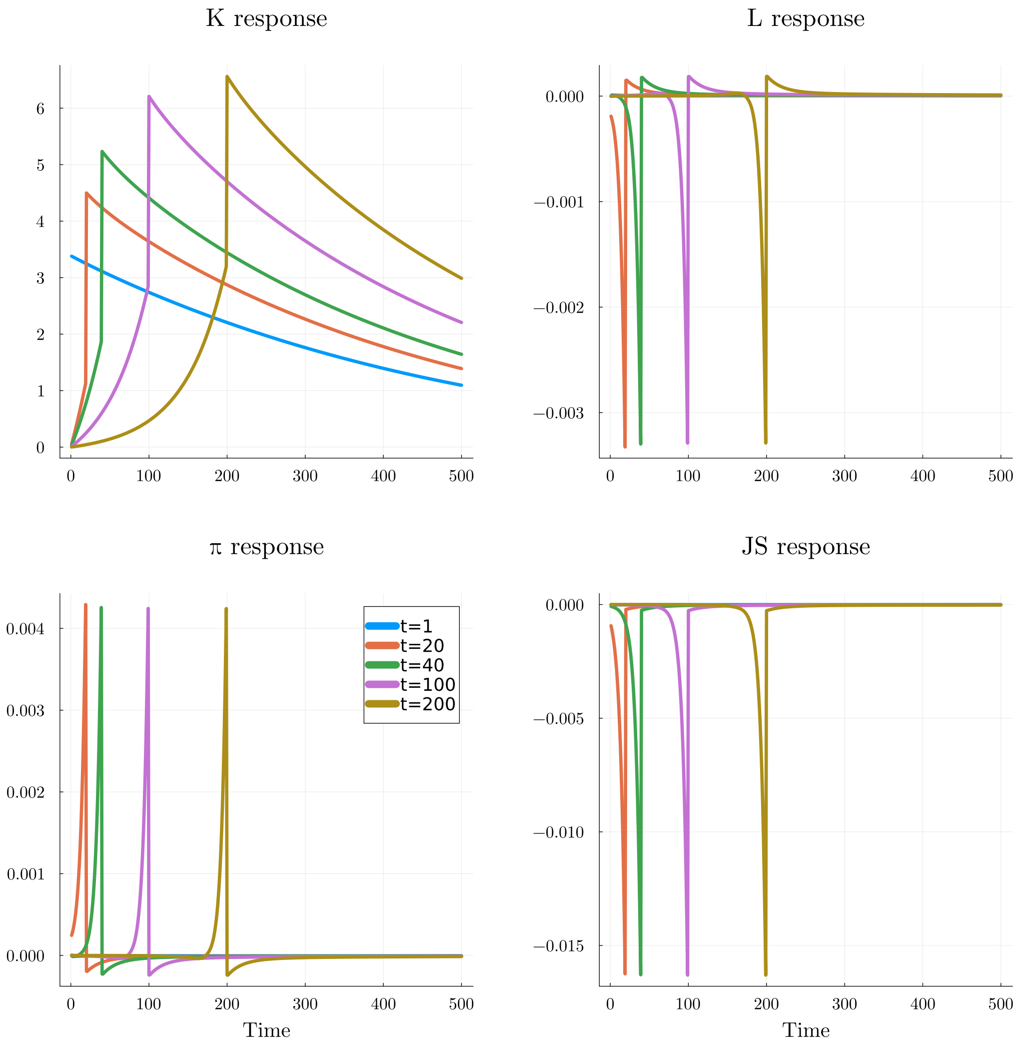

Figure 8: Columns of Joint Block SSJs with respect to r

6.4 Results

Figure 8 depicts columns of the resulting SSJs for several output variables of the Joint Block with respect to the input sequence $\{r\}$. Each line can be interpreted as the output variable's first order response to a pre-announced one time shock to the interest rate at the respective time t. The response of aggregate capital in the top left displays the characteristic "tent" patterns already noticed by Auclert et al. (2021), although the response is more persistent due to agents having relatively low marginal propensities to consume in the present calibration. Additionally, we notice that in response to an interest rate change, labor supply marginally decreases, profits slightly increase and the variable J^S decreases. The last two responses may seem contradictory at first, but are explained by the fact that r is not only the rate at which households save, but also the rate at which the model's firms discount future income. Thus, an increase in r decreases the net present value of a match, and in fact this decreased surplus also allows the firm to capture a slightly higher share of the match output during bargaining. The resulting slightly lower wages then

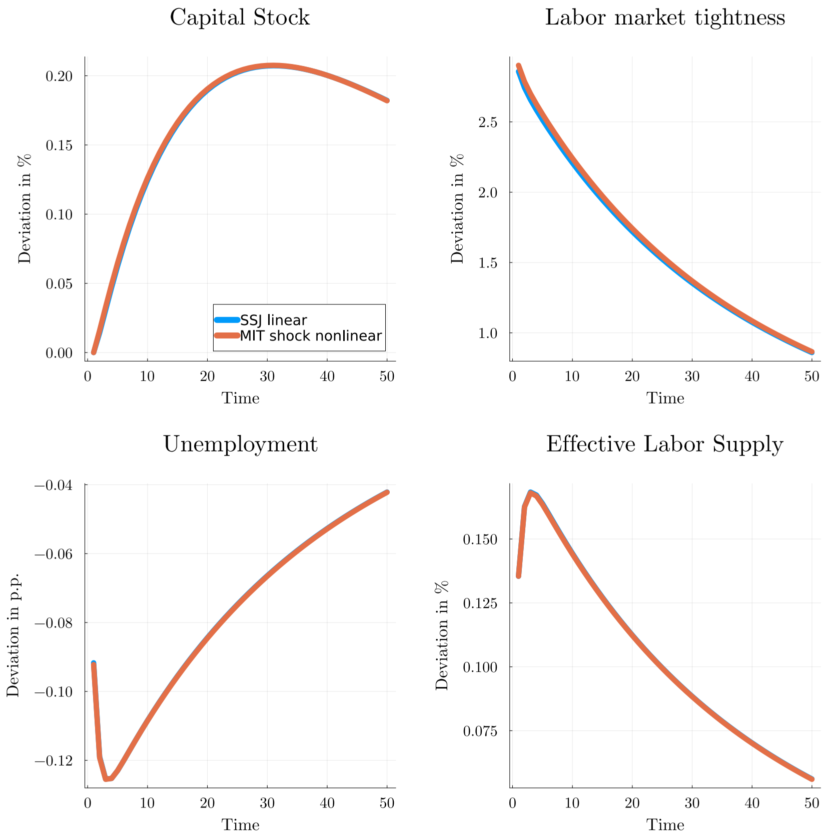

Figure 9: Aggregate responses to 25 bps TFP shock

cause a small decrease in intensive margin labor supply by households.

Finally, Figure 9 displays the corresponding linearized impulse response functions (IRFs) of some of the model's aggregate variables to a 25 basis point shock to aggregate TFP, which is assumed to follow an AR(1) process in logs

$$\log A_t = (1 - \rho_A) \log A_{ss} + \rho_A \log A_{t-1} + \epsilon_t$$

with $\rho_A = 0.95$. As Boppart et al. (2018) pointed out, for small shocks a model's linearized IRFs are approximately equal to its non-linear response following an unanticipated "MIT" shock of the same size. Hence, a natural check of the validity of the Auclert et al. (2021) method in this setting is a comparison with said non-linear response. It can easily be obtained by employing the SSJs for a quasi-Newton scheme and is also displayed in Figure 9 (orange line). Reassuringly, the non-linear perfect foresight responses are virtually indistinguishable from the linearized one for this relatively small shock.

Overall, the model's response looks as one would expect: The sudden and persistent increase of aggregate productivity raises investment and the aggregate capital stock. Labor market tightness jumps up immediately, causing a marked decline in unemployment, while aggregate labor supply increases even more as already employed workers also choose to work more hours.

7 Concluding Remarks

In this paper, I proposed a method to efficiently and reliably solve Bewley-type incomplete markets models with DMP labor market frictions and bilateral wage bargaining: The key innovation is to solve for the worker- and firm value functions jointly, which can be done using an extension of the Endogenous Grid Method. I demonstrated that the approach outperforms previous methods for a baseline model and is applicable to a variety of settings, including frameworks with endogenous separations or intensive margin labor supply. It is useful for solving stationary models as well as computing transition dynamics and linearized impulse response functions.

There should be no lack of interesting economic questions that can now be addressed using MIEGM. As mentioned above, a substantial literature has studied the effects of labor market policies on business cycle stabilization using BHA-DMP models, but relied on ad hoc wage rules to do so. Allowing wages to endogenously respond to such policies through explicitly modelled bargaining may open novel perspectives on their effectiveness.[19] MIEGM should also be useful to study how labor market frictions and -policies affect the long run income and wealth distribution, or, vice versa, how other redistributional policies may affect labor market outcomes. I plan to pursue some of these issues in ongoing and future research.

[19] In the context of a complete markets model, Mitman and Rabinovich (2019) argue wage responses to UI extensions to be important drivers of unemployment dynamics.

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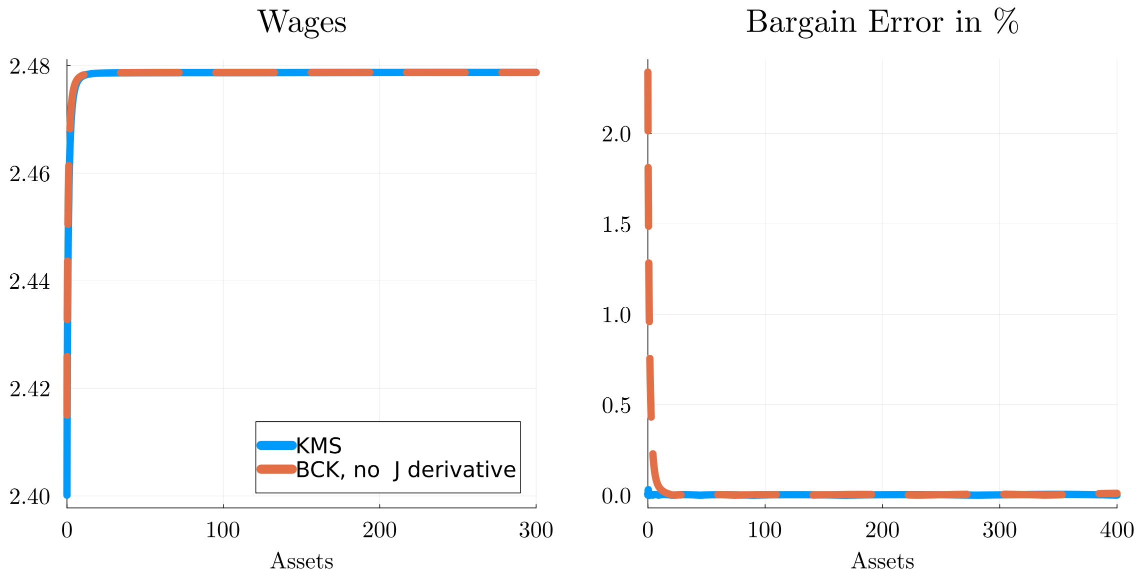

Figure 10: Effects of omitting $\frac{1-\delta}{1+r}\mathbb{E}\frac{\partial J}{\partial a'}\frac{\partial a'}{\partial w}$ term

Appendix

A Firm continuation value in the Bargaining FOC

As mentioned above, the original numerical implementation by Bils et al. (2011) considers an incorrect version of Nash FOC (5) not featuring the term $\frac{1-\delta}{1+r}\mathbb{E}\frac{\partial J}{\partial a'}\frac{\partial a'}{\partial w}$. The left panel of Figure 10 compares the resulting wage schedule (orange dashed line) with the (correct) version obtained using numerical minimization in the KMS manner (blue solid line), while the right panel presents the corresponding Bargaining Errors (based on the correct FOC).

We see that using the incorrect FOC results in too high wages for low asset holdings, which can result in Bargaining Errors of more than 2%. While it seems unlikely that correcting the mentioned omission would substantially affect the conclusions of the original Bils et al. (2011) paper, such errors might be more problematic for analyses as e.g. in Pizzo (2023), who studies the welfare effects of tax progressivity and appears to make a similar mistake.

B Effects of ζ_w choice

This Appendix provides additional analysis on how the convergence speed of the different algorithms depend on the of the wage updating parameter ζ_w, again in the context of the baseline KMS model. Figure 11 displays computation times for solving the partial equilibrium problem for different values of ζ_w. For the KMS model, MIEGM converges for essentially any ζ_w and is fastest at around $\zeta_w = 0.75$. The BCK and KMS methods, in

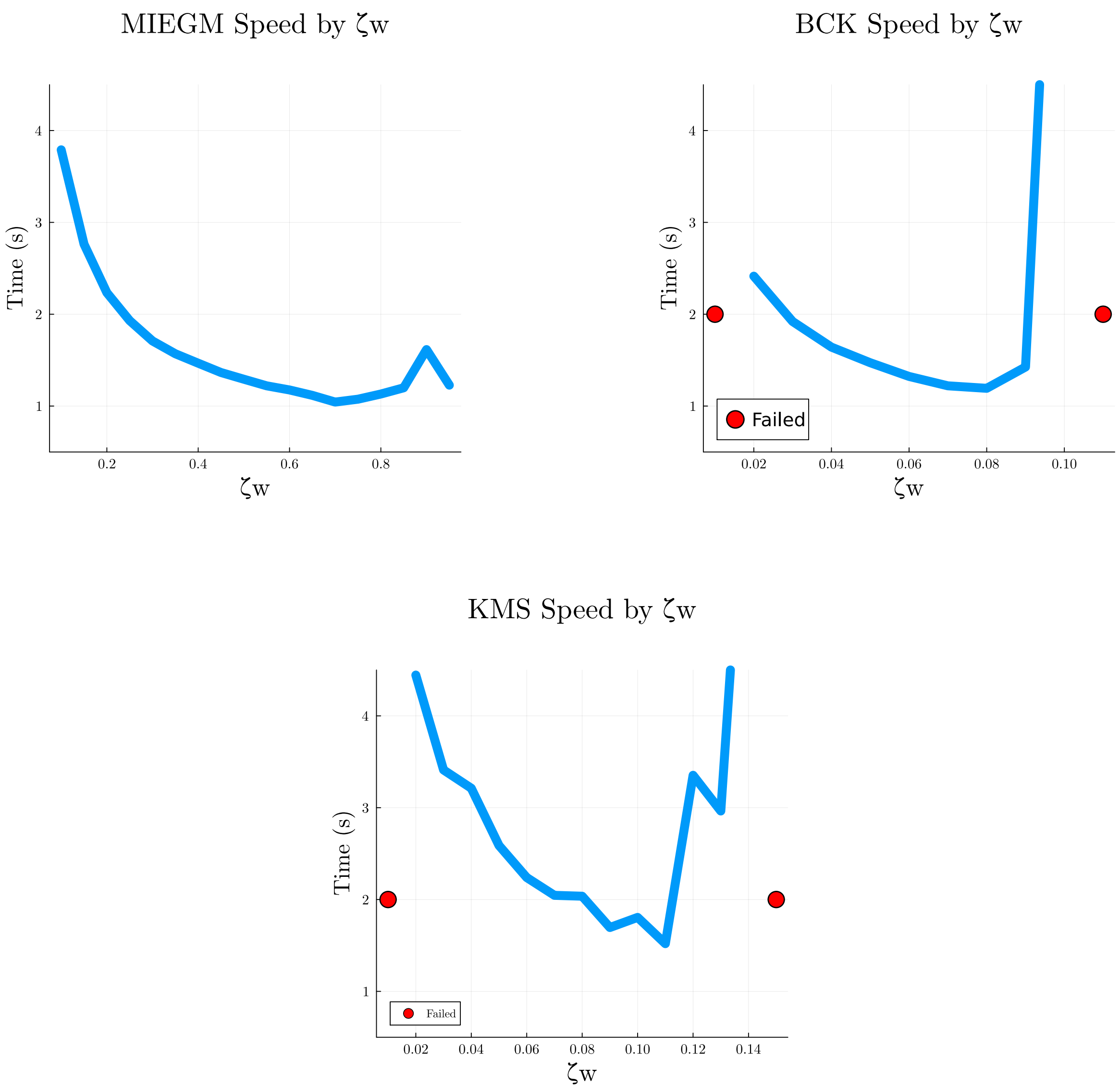

Figure 11: Algorithm speeds for different ζ_w values

turn, are fastest around $\zeta_w = 0.08$ or $\zeta_w = 0.1$ and fail to converge for values exceeding 0.11 or 0.14, respectively. Furthermore, their computational speed can deteriorate quickly for too high or low ζ_w values.

Ultimately, it should be noted that these results are specific to the model at hand and depend substantially on the bargaining power parameter γ: Intuitively, a relatively high worker bargaining power of $\gamma > 0.5$ induces the $\frac{1-\gamma}{\gamma}$ term in (6) to be < 1 so that it essentially becomes a "discount factor" stabilizing the updating process. In turn, for low γ values, the opposite ensues and iterating on (6) typically requires lower ζ_w values to ensure convergence.

C Details on Steady State experiment

This Appendix provides details on the steady state experiment conducted in Section 3.4. In particular, I search for k and θ such that both (10) and (9) hold at the implied r and vacancy filling rate, respectively. I start with initial guesses $k^0 = 74.0$ and $\theta^0 = 1.1$ and update according to

$$k^{n+1} = k^n + 0.001(\hat{k} - k^n) \ , \ \ \theta^{n+1} = \theta^n + 0.1(\hat{\theta} - \theta^n)$$

where $\hat{k}$ and $\hat{\theta}$ are given by

$$\hat{k} = \int f(a)da/(1-u) \ , \ \ \hat{\theta} = \left(\frac{\kappa}{A_m \int J(a'^u(a)) \frac{f_u(a)}{u} da} \right)^{-1/\eta}$$

i.e. the respective values implied by household choices and firm value functions at current r and p^{ue}. I deliberately consider starting values that are somewhat "off" to mimic the realistic case in which one does not have very good ex-ante knowledge about the model at hand. I consider fairly simple updating rules for similar reasons. The low updating parameter for k was chosen to ensure convergence. The tolerance criteria are $\frac{k^n - \hat{k}}{k^n} < 1e-4$ and $\frac{\theta^n - \hat{\theta}}{\theta^n} < 1e-4$.

For all algorithms, the scheme takes between 40 and 50 iterations to converge. The steady state equilibrium features $k = 72.1$ and $\theta = 1.0$, implying $r + \delta = 0.015$ and $p^{ue} = 0.6$ as targeted by the KMS calibration.

D KMS model: Alternative wage approximation

When analyzing the KMS model in the main text, I use Chebyshev regression to approximate the wage function off-grid. Here, I wish to briefly ascertain that the respective methods also work when using cubic splines, a more common function approximation method. However, as mentioned in Section 2.5, these are prone to generate non-concavities in high-curvature part of the wage function: this makes it approximated

	MIEGM	BCK (EGM)	KMS (EGM)
Partial eq.	0.727s	1.29s	1.414s
GE Steady State	15.37s	46.23s	32.28s

Note: Reported time is the best out of 3 runs. For additional details on the Steady State experiment, see Appendix C.

Table 5: Algorithm computing times: Alternative wage approximation

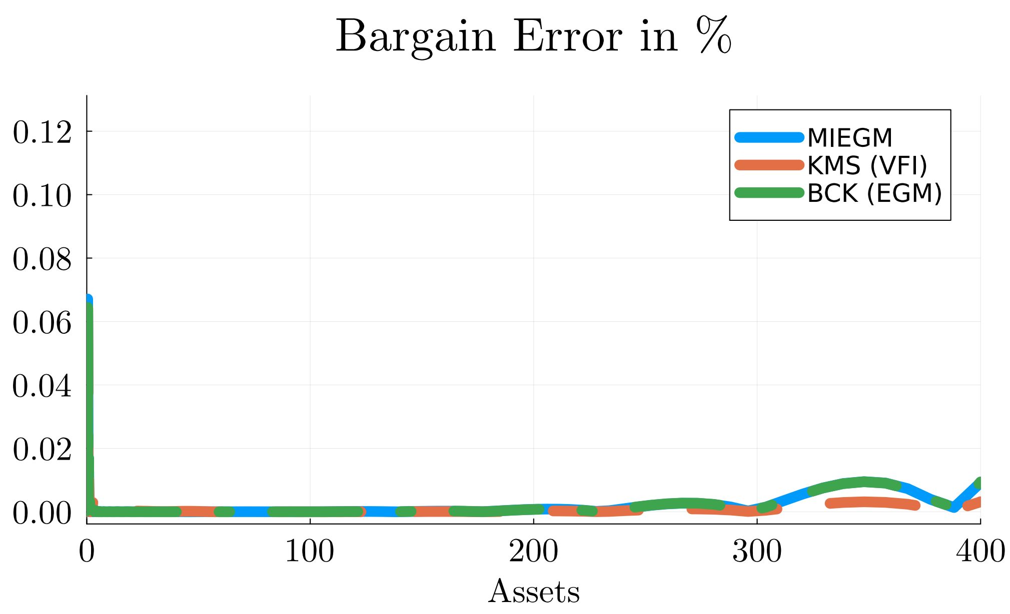

Figure 12: Wage accuracy check for spline approximation

derivative non-monotone and causes errors in the EGM steps of the various algorithms. To counteract this, I use relatively small $\mathcal{A}_w$ with $n_w = 25$, which is numerically more stable and ensures convergence of all algorithms. For example, I found that a log-spaced grid of, e.g., size $n_w = 50$, would result in errors or non-convergence for all of the considered methods (including the VFI-based ones).[20]

In Table 5, we see that using the alternative scheme does not substantially change the absolute or relative computational efficiency of the different methods: MIEGM and KMS benefit somewhat. This, however, features a tradeoff. Figure 12 displays the corresponding bargaining errors off the $\mathcal{A}_w$ grid, equivalent to Figure 2 in the main text. In particular, we get larger (but overall still small) errors at low worker wealth.

[20]While the original KMS paper uses cubic splines with a greater number of n_w, the authors seem to use a linearily spaced grid that effectively also features very few points in the wage function's high-curvature region.

E Applying the fake news algorithm to the Nakajima (2012) model

This Appendix explains how I implement a variant of the "fake news" algorithm of Auclert et al. (2021) to obtain the Sequence Space Jacobian of the Nakajima (2012) model's Joint Block truncated at some $\mathcal{T}$. My description focuses on the Jacobian with respect to the interest rate r, but the procedure is analogous for the other input sequences.
For that purposes, let me first define the set of "heterogeneous agent" inputs from period $t+1$ that will be necessary to iterate backwards using MIEGM:[21]

$$HA_{t+1} := \left[w_{t+1}(s,a), l_{t+1}(s,a), a'_{t+1}(s,a), V_{t+1}(s,a), J_{t+1}(s,a)\right] \quad .$$

MIEGM can thus be treated as function uses inputs HA_{t+1} and aggregate inputs p^{UE}_{t+1},r_{t+1},r_t, p_t as well as T_t to compute the corresponding inputs HA_t for period t, i.e.

$$HA_t = MIEGM\left(HA_{t+1}, p^{UE}_{t+1}, r_{t+1}, r_t, p_t, T\right) \quad . \tag{24}$$

The model's endogenous distribution changes within the period. If the initial distribution in t is $D_t(e,s,a)$, there are first employment transitions taking place at the beginning of period t and transitions in a and s at the end of the period. In turn, the discretized evolution of $\mathbf{D}_t$ (treated as a vector) evolves according to

$$\mathbf{D}_{t+1} = \underbrace{\Lambda'_{sa}(\mathbf{a'}_t)\Lambda'_e(p^{UE_t})}_{:=\Lambda'_t}\mathbf{D}_t$$

where $\Lambda'_{sa,t}$ is a transition matrix for (s,a) (depending on time t savings choices) and $\Lambda'_{e,t}$ is a transition matrix for employment status (depending on time t job finding rate). Also note that within a period, different distributions will be relevant for the different outputs of the Joint Block. For S_t and J^S_t, it will be $\mathbf{D}_t$, while for all other variables it is $\Lambda'_{e,t}\mathbf{D}_t$ - aggregate savings, labor supply decisions etc. only take place after labor market transitions took place.
In the description of their algorithm, Auclert et al. (2021) treat a "macro"-output Y of a heterogeneous agents block as a combination of the distribution $\mathbf{D}$ and micro outputs y as $Y = y' * \mathbf{D}$. In the given model, the relevant relations for the outputs as on Figure 7 are

- $A_{t+1} = (\mathbf{a}'_t(e,s,a))' * (\Lambda'_{e,t}\mathbf{D}_t)$
- $L_t = (\mathbf{1}(e) \odot (\mathbf{l}_t(s,a) \odot \mathbf{s}(s)))' * (\Lambda'_{e,t}\mathbf{D}_t)$
- $S_t = (\mathbf{1}(u) + \mathbf{1}(e)\delta)' * \mathbf{D}_t$
- $J^S_t = (\mathbf{J}_t(s,a) \odot (\mathbf{1}(u) + \mathbf{1}(e)\delta))' * \mathbf{D}_t$
- $\pi_t = p_t(\mathbf{1}(e) \odot (1 - \mathbf{w}_t(s,a)) \odot \mathbf{l}_t(s,a) \odot \mathbf{s}(s))' * (\Lambda'_{e,t}\mathbf{D}_t)$

[21] In principle, the value functions J and V alone characterize all of them, but using only these would require costly numerical optimization to obtain the rest.

- $gs_t = (\mathbf{1}(e) \odot \tau p_t \mathbf{w}_t(s,a) \odot \mathbf{l}_t(s,a) \odot \mathbf{s}(s) - \mathbf{1}(u) \odot \mathbf{b}(s,a)) * (\Lambda'_{e,t}\mathbf{D}_t)$

where $\mathbf{1}(u)$ and $\mathbf{1}(e)$ are indicator vectors for (un)employment and $\odot$ denotes the element-wise (Hadamard) product. For the expressions above, a variable $\mathbf{y}(s,a)$ denotes a vector stacked so that $y(s,a)$ is at the same place as any entry on $\mathbf{D}$ corresponding to these skill- and wealth levels. The equivalent applies to $\mathbf{y}(e,s,a)$ and $\mathbf{y}(s)$ terms.
After having gone through these formalities, it is now possible to describe how to obtain the SSJ for the Joint Block with respect to r.

1. Define a function F that does the following:

   (a) Taking as scalar input dr (a perturbation of the steady state interest rate), it iterates on (24) backwards from $t = \mathcal{T}+1$ to $t = 0$, using $r_{\mathcal{T}} = r_{ss} + dr$, $r_t = r_{ss} \forall t \neq \mathcal{T}$ as well as $p_t^{UE} = p_{ss}^{UE}$, $p_t = p_{ss}$ and $T_t = T_{ss} \;\; \forall t$.

   (b) For every $t \leq \mathcal{T}$, it saves the following function outputs:

   - $A_0^{\mathcal{T}-t} = (\mathbf{a}'_t(e,s,a))' * (\Lambda'_e(p^{UE_t})\mathbf{D}_{ss})$
   - $L_0^{\mathcal{T}-t} = (\mathbf{1}(e) \odot (\mathbf{l}_t(s,a) \odot \mathbf{s}(s)))' * (\Lambda'_e(p^{UE_t})\mathbf{D}_{ss})$
   - $\vdots$
   - $gs_0^{\mathcal{T}-t} = (\mathbf{1}(e) \odot \tau p_t \mathbf{w}_t(s,a) \odot \mathbf{l}_t(s,a) \odot \mathbf{s}(s) - \mathbf{1}(u) \odot \mathbf{b}(s,a)) * (\Lambda'_e(p^{UE_t})\mathbf{D}_{ss})$
   - $D_0^{\mathcal{T}-t} = \Lambda'_{sa}(\mathbf{a'}_t)\Lambda'_e(p^{UE_t})\mathbf{D}_{ss}$

   where $\mathbf{a}'_t(e,s,a)$, $\mathbf{l}_t(s,a)$ etc. are obtained from HM_t. Note that the employment transition matrix using p_t^{UE} (instead of p_{ss}^{UE}) is used to compute the objects above. This follows Appendix A.1 in Auclert et al. (2021).
   Optional: Before proceeding to $t-1$, it conducts the re-centering step as described in Appendix C of Auclert et al. (2021).

2. Differentiate all output terms of that function with respect to dr using Forward-mode automatic differentiation. This only requires a single function evaluation and provides the $\mathcal{Y}$ and $\mathcal{D}$ terms as defined by Auclert et al. (2021).[22]

3. Afterwards, proceed as described in Section 3.2 of Auclert et al. (2021).

[22]This avoids dealing with several tedious product-rule terms one would encounter if one were to compute these objects with the finite difference approach as described in the original paper (as the block outputs feature products of different variables depending on the shock).